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THE GROUND OF ARTES.

THROUGHOUT its life-time, the Mathematical Association has been much concerned with textbooks, from preparing " substitutes for Euclid " in 1870 to discussing possibilities of textbooks for a unified school course in 1947 ; and among its members it has been fortunate enough to number many of the famous textbook authors of the day. It is therefore appropriate to include in our series of illustrations some examples of earlier " best sellers ".

The books of Robert Recorde have a long history, from their first appearance in the sixteenth century, and it is possible to trace a continuous line of their influence even as far as the present day. Reference may be made to the relevant portions of Mr. W. F. Bushell's Presidential Address, in the *Gazette* for May, 1947.

Recorde was born at Tenby about 1510. He studied at Oxford, migrated to Cambridge in order, it is said, to read mathematics and medicine, returned to lecture at Oxford, and finally moved to London. His *Ground of Artes*, the first popular arithmetic in English, was first published about 1540, and ran into many editions. In 1582 the notable John Dee prepared a revised version, and editions went on appearing through the next century, the last known being one of 1699. Its popularity was not confined to these islands, for it is mentioned as being exported to the English colonies in America, and being used, for example, in Massachusetts in the middle of the seventeenth century. The dialogue form, now merely quaint, may have had advantages in its time ; another merit is the thoroughly practical tone, on which D. E. Smith comments : " Never was there a better opportunity for a commercial arithmetic, and never was opportunity more successfully met ".

For the present plate, we are indebted to the British Museum Library and Photographic Department.

ISOGONAL CONJUGATES.

A NEW APPROACH TO CERTAIN GEOMETRICAL THEOREMS AND TO A GENERAL THEORY OF CONICS.

By H. E. Piggott and A. Steiner.*

1.0. ABC is a triangle; BX, CX' are equal arcs of the circumcircle taken in opposite senses from B and C. AX, AX' are then said to be isogonally conjugate with respect to AB and AC. (The usual textbook definition is to say that the angles BAX, CAX' are equal. The definition by equal arcs is used throughout this paper both because it has practical advantages and because it leads easily to theoretical development.)

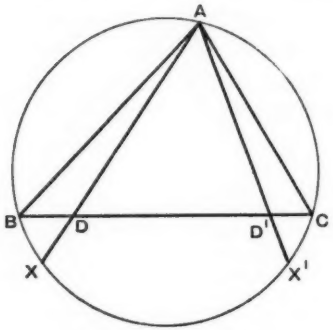


Fig. 1.

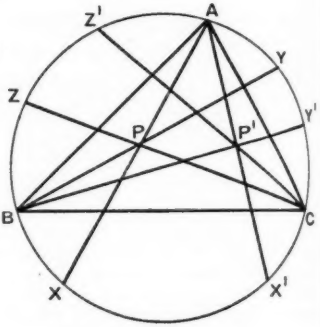


Fig. 2.

1.1. It is shown in standard textbooks that if AX, BY, CZ are concurrent in P, then AX', BY', CZ' are concurrent in, say, P'. P and P' are called isogonally conjugate points or anticentres.

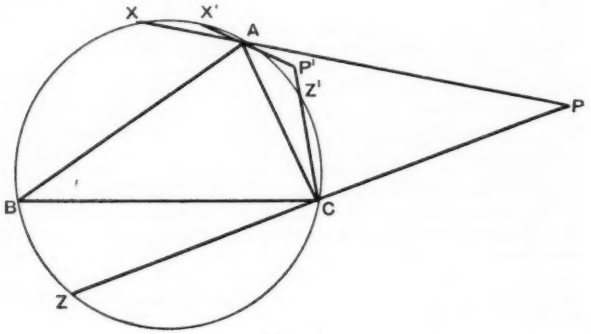


Fig. 3.

* Although this paper, an account of which was given at the Annual Meeting, 25th April, 1946, was constructed in collaboration, the first-named was responsible for the final shape of the sections 1–4 on pure geometry, and the latter for the development and shape of the analytical sections, 5–8. In this printed form, the two portions have, therefore, been separately written and signed.

Given a point P, to find its anticentre it is necessary to draw two lines only, *e.g.* AX and BY; then by stepping off CX' equal to BX and AY' equal to CY, the cut of AX' and BY' gives P'.

1.2. If P lies inside the triangle, so also does P'. If P lies outside the triangle, so also does P'. Fig. 3 shows this case.

1.3. If P is any point on the circumference of the circumcircle, except A, B or C, then X, Y, Z are all at P. It is easy to see by angle properties that AX', BY', CZ' are all parallel. Thus P' is at infinity in the direction given by AX', BY', CZ'.

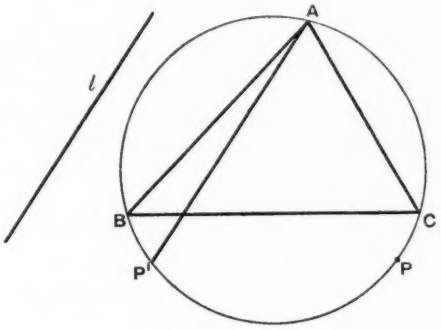


FIG. 4.

The converse construction will be needed in the sequel. l is a given straight line: it is required to find the point which is the isogonal conjugate of the point at infinity on l. Draw AP' parallel to l meeting the circumcircle in P'. Step off CP equal to BP' in the reverse sense. Then P is the required point.

1.4. The isogonal conjugate of any point on BC or BC produced, except B and C, is A.

There appears at first sight to be some uncertainty about the anticentres of A, B, C. There needs to be a more precise definition. As the development which follows deals mainly with loci, it is better to go back to the definition of isogonally conjugate lines in 1.0. AX' may be considered as the locus of the anticentre of P as P moves along the line AX. If P is at X, P' is at infinity along $X'A$ produced. As P moves upwards to D, where AX cuts BC, P' moves down to A. The two points now traverse the inside of the triangle in opposite directions, so that when P has reached A, P' is at D' where AX' cuts BC. AX is undefined in direction, so D' may take any position in BC or BC produced, except B and C. The reciprocal relation is thus established.

1.5. It is easily shown that if P, P' are anticentres and if PL, $P'L'$ are perpendicular to BC, PM, $P'M'$ perpendicular to CA and PN, $P'N'$ perpendicular to AB, then $PL \cdot P'L' = PM \cdot P'M' = PN \cdot P'N'$.

Thus if trilinear coordinates of P are (x, y, z), those of P' are $(1/x, 1/y, 1/z)$. The trilinear coordinates of I, the incentre, are $(1, 1, 1)$. Hence I is its own isogonal conjugate. The same is true of the e-centres I_1, I_2, I_3. It is instructive to consider, on the lines of 1.4, the movements of P, P', starting them off in different directions from the point I, along the line AII_1.

The circumcentre O and the orthocentre H are anticentres, their trilinear coordinates being respectively ($\cos A$, $\cos B$, $\cos C$) and ($\sec A$, $\sec B$, $\sec C$). Two other notable pairs will now be considered.

2.0. If BX, CX are drawn through B and C, respectively parallel to AC, AB to intersect in X, and if Y and Z are similarly constructed, it is clear that AX, BY, CZ are medians of the triangle and intersect in G, the centroid.

2.1. If BX, CX are drawn tangential to the circumcircle at B and C and intersect at X, AX cutting BC at S (see Fig. 5), we have

$$\begin{aligned} BS : CS &= \triangle ABX : \triangle ACX \\ &= \tfrac{1}{2}BX \,.\, c \,.\, \sin C : \tfrac{1}{2}CX \,.\, b \,.\, \sin B \\ &= c^2 : b^2. \end{aligned}$$

Hence, by Ceva's theorem, AX, BY, CZ are concurrent. Draw DXE parallel to the tangent to the circle at A, to cut AB, AC produced in D and E. Then $\angle D = \angle C$, and $\angle E = \angle B$. So $DX = BX = CX = EX$; thus DE is antiparallel to BC with respect to AB, AC, and is bisected at X. So AX bisects all lines drawn antiparallel to BC and terminated by AB, AC. AX is called a symmedian line, and K, the point of concurrence of AX, BY, CZ, is called the symmedian point of the triangle. K plays a most important part in the analytical development to follow.

2.2. Through K draw the three lines MKL', $M'KN$, LKN' antiparallel respectively to AB, BC, CA. It is clear, from the preceding, that KL, KL', KM, KM', KN, KN' are all equal. If ρ is the radius of the circle of centre K passing through the six points L, L', M, M', N, N', then $LL' = 2\rho \cos A$. So the circle cuts off intercepts from the sides respectively proportional to $\cos A$, $\cos B$, $\cos C$. From this property it is known as the Cosine circle of the triangle.

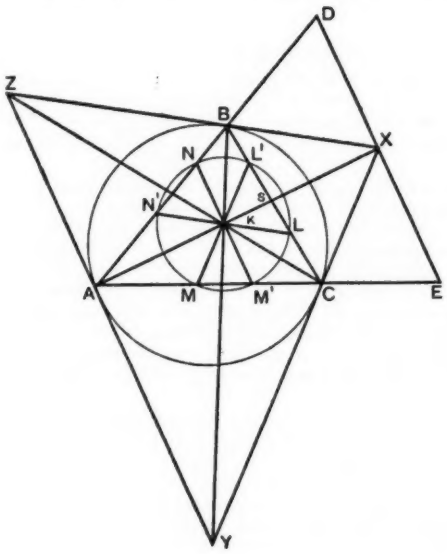


FIG. 5.

The perpendiculars from K to the sides are respectively $\rho \sin A$, $\rho \sin B$, $\rho \sin C$. Hence the trilinear coordinates of K are (a, b, c). Now it is easy to see that the trilinear coordinates of the centroid G are $(1/a, 1/b, 1/c)$. Hence G and K are isogonal conjugates.

2.3. In Fig. 5, $L'M$ is a diameter of the cosine circle, so the angle MLL' is a right angle. Thus $CL = ML \cot C = 2\rho \sin A \cot C$. Thus

$$\begin{aligned} a &= LL' + L'B + L'C \\ &= 2\rho\,(\cos A + \sin A \cot B + \sin A \cot C) \end{aligned}$$

whence $$R = \rho\,(\cot A + \cot B + \cot C).$$

3.0. Now combine the constructions of 2.0 and 2.1, *i.e.* draw BX parallel to AC and CX antiparallel to AB.

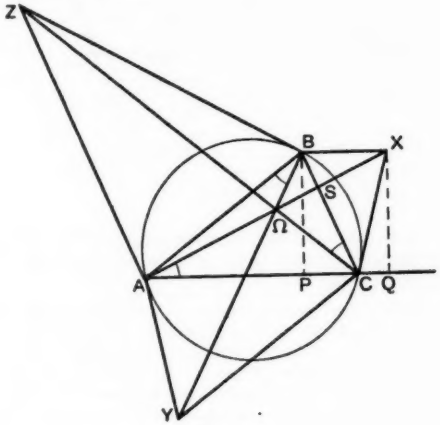


FIG. 6.

Then

$$\begin{aligned} BS : SC &= \triangle ABX : \triangle ACX \\ &= \tfrac{1}{2}BX \,.\, c \,.\, \sin A : \tfrac{1}{2}CX \,.\, b \,.\, \sin B \\ &= \sin A \,.\, c \,.\, \sin A : \sin C \,.\, b \,.\, \sin B \\ &= a^2 : b^2. \end{aligned}$$

So, again by Ceva's theorem, AX, BY, CZ are concurrent.

Draw BP, XQ perpendicular to AC and call the length of each unity. Then

$$\begin{aligned} \cot XAC = AQ &= AP + CQ + PC \\ &= \cot A + \cot B + \cot C. \end{aligned}$$

Hence the angles XAC, YBA, ZCB are each equal to ω, where

$$\cot \omega = \cot A + \cot B + \cot C.$$

The angle ω is called the Brocard angle of the triangle, and Ω, the point of concurrence of AX, BY, CZ, is called a Brocard point. If the construction is performed " the other way round ", with the parallel through C and the antiparallel through B we arrive at the second Brocard point Ω'. Ω and Ω' are isogonal conjugates, their trilinear coordinates being $(c/b, a/c, b/a)$ and $(b/c, c/a, a/b)$.

3.1. We can now obtain a neater expression for ρ, the radius of the cosine circle. From 2.3, $\rho = R \tan \omega$ and the perpendicular from XK to the side BC is $\frac{1}{2}a \tan \omega$.

3.2. In Fig. 7 the perpendicular bisector of AC meets the line AX constructed as in 3.0 in P and PO meets AC at B'; $B'P = \frac{1}{2}b \tan \omega$. So PK parallel to AC meets the symmedian line AX' in K. Hence the circle on OK as diameter

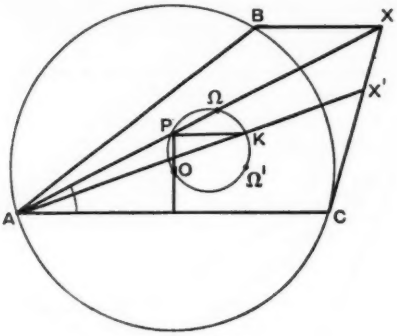


FIG. 7.

passes through P and through the symmetrically constructed points Q and R. Angle properties show that this circle also passes through Ω and Ω'. It is known as the Brocard circle.

3.3. If through K lines are drawn parallel to the sides of the triangle, LKN', $L'KM$, NKM', it can be shown without difficulty that the six points L, L', M, M', N, N' are concyclic and that the circle on which they lie is concentric with the Brocard circle. Its radius is $\frac{1}{2}R \sec \omega$. It is known as the Lemoine circle or the Triplicate Ratio circle. The latter name arises from the fact that it can be shown to cut off from the sides of the triangle intercepts which are in the ratio $a^3 : b^3 : c^3$.

This linkage between K and the Brocard angle and points gives rise to a number of interesting properties. Enough has been given to serve as a foundation for the analytical work to follow.

4.0. If, in Fig. 1, the lines AX, AX' cut off equal intercepts in opposite senses from the line BC (that is, using the chord instead of the arc of the circumcircle) so that $BD = CD'$, then AX, AX' are said to be isotomically conjugate with respect to AB and AC. It can be shown, on the same lines as for isogonal conjugates, that if AX, BY, CZ are concurrent in a point P, then AX', BY', CZ' are concurrent in a point P' called the isotomic conjugate of P with respect to the triangle of reference. It can be further shown on the lines of 1.5 that if the *areal* coordinates of P are (x, y, z) then those of P' are $(1/x, 1/y, 1/z)$. G is now its own isotomic conjugate (1, 1, 1), so taking the place of I in the other system. It will later be shown that the isotomic conjugates of points at infinity all lie on Steiner's ellipse * $yz + zx + xy = 0$. This therefore takes the place of the circumcircle in the other system. It is now instructive to trace the movement of P' along AX' as P moves along AX, as was done in 1.4 for isogonals.

H. E. P.

* Steiner's ellipse for our purposes is defined by the areal equation $yz + zx + xy = 0$. But we may remind readers that it is the ellipse which passes through the vertices A, B, C of a triangle such that the tangent to it at A is parallel to BC. Obviously its centre is the point G.

5. Conics and isogonal conjugates in trilinears.

5.0. Let a straight line m with equation $fx+gy+hz=0$ cut the sides of the triangle of reference in X, Y, Z. The isogonal conjugates of these points are A, B, C (1.4). The isogonal conjugate of any other point P (x, y, z) on m is a point P' whose coordinates are $(1/x, 1/y, 1/z)$, by (1.5). Thus P' lies on a conic μ, whose equation is $fyz+gzx+hxy=0$. This conic passes through the vertices of the triangle of reference and we shall refer to it as a *circumconic*. Conversely, the isogonal conjugates of A, B, C are the points X, Y, Z (1.4) and the isogonal conjugate of any other point on μ lies on m. That is, the locus of the isogonal conjugate of a point on a line m is a circumconic μ, and conversely. We shall call m the *generating line*, and say that m *generates* μ.

5.1. The isogonal conjugate of a point at infinity is on the circumcircle, and conversely (1.3). That is, the line at infinity generates the circumcircle. In future, we shall write simply " circle " for the circumcircle.

5.2. Two isogonally conjugate lines will be denoted by u, u'. Now let u be a line through A cutting the circle again in S (Fig. 8). Step off the equal arcs

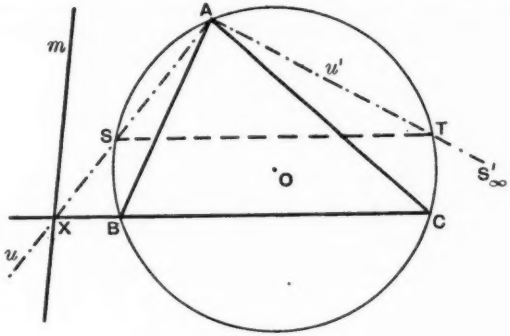


FIG. 8.

BS, TC. By 1.0, u' is the line AT. The isogonal conjugate S' of the point S is on u' (1.1) and on the line at infinity, since S is on the circle (5.1). Hence S' is the point at infinity on u', the line AT.

5.3. If D is a point of the circumconic μ, then the isogonal conjugate D' is a point on a straight line m (5.0). Thus a pencil of straight lines through a point D' generates a family of conics through the four-point A, B, C, D.

5.4. If D is a point on the circle, then D' is a point at infinity (5.1), that is, D' is defined by a set of parallel lines. Thus in this case (5.3) shows that a system of parallel lines generates a family of conics through four concyclic points. It is well known (see, for instance, Robson, *Analytical Geometry*, I, 16.81) that conics of such a system have parallel axes. Thus, a system of parallel straight lines generates a system of conics with parallel axes.

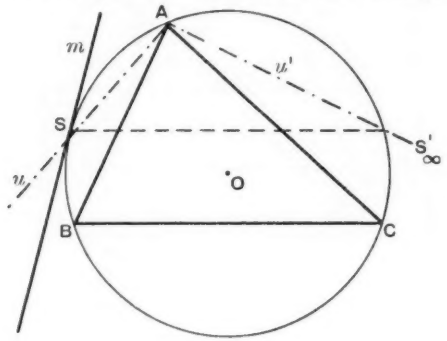
5.5. Let the generating line m cut the circle in S and T. Then S' and T' are on μ (5.0) and also on the line at infinity (5.1). Hence S' and T' are the points at infinity on the circumconic μ. Thus this conic is an ellipse, parabola or hyperbola according as S', T' are imaginary, real and coincident or real and distinct. That is, μ is an ellipse, parabola or hyperbola according as its generating line is outside, touches or cuts the circle.

5.6. Let u cut BC in X (Fig. 8). Then X' is the point A (1.4). The isogonal conjugate of A (considered as a point of u) is any point on BC (1.4). The isogonal conjugate of any other point on u is on u' (1.1). Thus the conic generated by u is the pair of lines BC, u'.

5.7. Let the generating line m cut BC in X (Fig. 8), and take u to be AX. The two conics generated by the lines m and u pass through A, B, C, X' (5.3). But X' is the point A (1.4). Thus these two conics touch at A. Now by (5.6) the conic generated by u is the line-pair u', BC. Thus u' is the tangent to μ at A. The tangents at B and C can be constructed in the same way.

5.8. If the tangents to μ at B, C meet at D, the tangents to μ at C, A meet at E and the tangents to μ at A, B meet at F, then the lines which join D to the midpoint of BC, E to the midpoint of CA, F to the midpoint of AB are concurrent in the centre of the conic μ. For, tangents at points B, C on a conic meet at a point on the diameter of the conic which bisects chords parallel to BC (Robson, *loc. cit.*, 12.54). Thus D and the midpoint of BC are two points on the same diameter of the conic ; hence the result.

5.9. Let m touch the circle at S (Fig. 9). Then by (5.5), μ is a parabola. Take the line u to be AS. Then S' is on u', but by (5.5) S' is also the point at infinity on μ. Thus u' gives the direction of the axis of the parabola.



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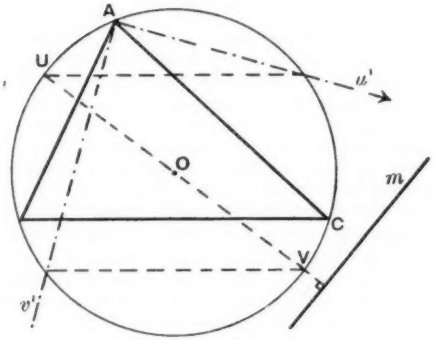


FIG. 10.

5.10. Let m be any generating line (Fig. 10), and let the perpendicular to m from the circumcentre O cut the circle in U and V. Take the lines u and v to be AU, AV. Lines parallel to m generate a family of conics with parallel axes (5.4) and so the lines through U and V parallel to m generate parabolas (5.9) whose axes are parallel to the axes of the family. By (5.9) the axes of these parabolas have directions given by u', v'. Hence the directions of the axes of the conic μ are given by u', v'.

5.11. Let m cut the circle in two real points S and T (Fig. 11). Then by (5.5) μ is a hyperbola. Take u and v to be AS, AT. Then the points at infinity on

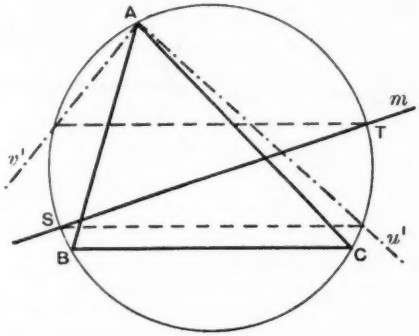


FIG. 11.

μ are on u' and v' (5.2). Thus u' and v' give the directions of the asymptotes of μ.

5.12. If m passes through the circumcentre O, so that ST is a diameter of the circle, and if u, v are AS, AT, then u' and v' are at right angles (Fig. 12). But

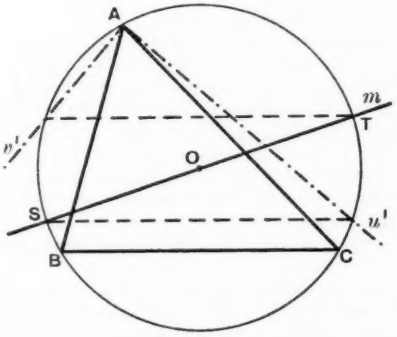


FIG. 12.

these give the directions of the asymptotes of the conic μ generated by m (5.11). Hence a line through the circumcentre generates a rectangular hyperbola.

5.13. Let u, u' be isogonally conjugate lines through A, cutting the circle again in S and T, so that ST is parallel to BC (Fig. 13). By obvious angle

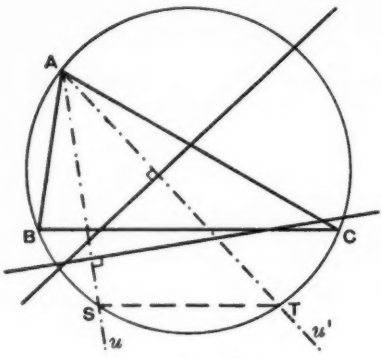


FIG. 13.

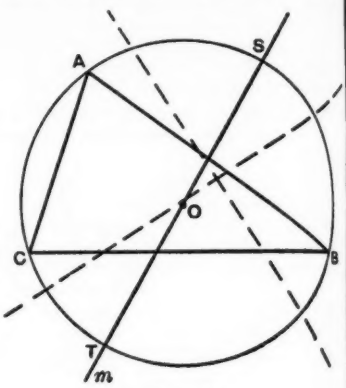


FIG. 14.

properties, u and u' are perpendicular to the Simson lines of T and S respectively (see, for instance, Durell, *Modern Geometry*, p. 69). Thus any construction based on isogonally conjugate lines, such as those for the tangents at A, B, C, for the axes, and for the asymptotic directions, can be performed as constructions with the corresponding Simson lines. Thus the axes of the conic generated by the line m in Fig. 10 are parallel to the Simson lines of U and V, and the asymptotes of this conic (Fig. 11) are parallel to the Simson lines of S and T. It should be particularly noted that in the case of a rectangular hyperbola (5.12) the Simson lines of S and T (the points in which m cuts the circle, m being now a diameter of the circle) are the actual asymptotes of μ (Fig. 14).

6. *Conics and isotomic conjugates in areals.*

6.0. In areal coordinates, $P(x, y, z)$ and $P'(1/x, 1/y, 1/z)$ are isotomic conjugate points (4.0). The argument of 5.0 remains valid for isotomic conjugates; that is, the locus of the isotomic conjugates of a point on a given straight line m is a circumconic μ, and as before we say that m generates μ.

6.1. From the relation between the coordinates of P and P', it is clear that the straight line $x+y+z=0$ generates the conic $yz+zx+xy=0$, Steiner's ellipse. But this straight line is the line at infinity in areal coordinates. Thus in this system, 5.1 becomes: the line at infinity generates Steiner's ellipse.

6.2. Let u be a line through A (Fig. 15), cutting BC at X. Step off a segment CY equal to XB. Then u', the line isotomically conjugate to u, is the line AY (4.0).

6.3. Constructing the isotomic conjugate u' of a line u by (6.2) and reading "Steiner's ellipse" (6.1) for "the circle", all constructions of § 5, for tangents (5.7), centre (5.8), axes (5.9 and 5.10), and asymptotes (5.11) can be performed in the areal system. Note that since the circumcentre is the *isogonal* conjugate of the orthocentre (1.5), the new enunciation of 5.12 is: a line through the isotomic conjugate of the orthocentre generates a rectangular hyperbola.

6.4. The results of the preceding section can be generalised to fit any system

of homogeneous coordinates. This leads, as I hope to show later, to a new geometrical interpretation of homogeneous coordinates.

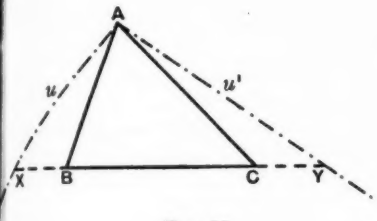


FIG. 15.

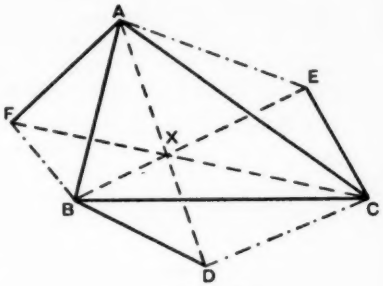


FIG. 16.

7. *The Brocard figures in trilinears.*

7.0. We write M for the point whose trilinear coordinates are (f, g, h) and μ for the corresponding circumconic (f, g, h), that is, the conic whose equation is $fyz+gzx+hxy=0$, μ' for the conic $(1/f, 1/g, 1/h)$ and $\triangle\mu$ for the triangle formed by the tangents to μ at A, B, C.

7.1. Let AF, BD, CE and AE, BF, CD (Fig. 16) be tangents to the circumconics $\lambda(p, q, r)$ and $\nu(s, t, u)$ respectively. Now AF is $qz+ry=0$, BF is $sz+ux=0$. Any line through F is therefore $qz+ry+\lambda(sz+ux)=0$ and this goes through C if $q+\lambda s=0$. Hence CF is $rsy-qux=0$, with obvious symmetrical equations for BE, AD and an obvious point of concurrence X with coordinates $(1/qu, 1/rs, 1/pt)$.

7.2. Let $\lambda=\mu=\nu$. Then X becomes the point $(1/gh, 1/hf, 1/fg)$, that is, (f, g, h), which is the point M (7.0), and the hexagon of (7.1) degenerates into $\triangle\mu$. Thus, the lines joining the vertices of $\triangle\mu$ to the opposite vertices of the reference triangle are concurrent in M.

7.3. Writing now a, b, c for f, g, h in (7.2), M becomes the point (a, b, c), that is, the symmedian point K (2.2), and μ becomes the conic (a, b, c). But by the definition of the symmedian point (2.1), when M is the symmedian point, μ is the circumcircle. Hence the equation of the circumcircle is $ayz+bxz+cxy=0$, and this is generated by the line $ax+by+cz=0$. But the circumcircle is generated by the line at infinity (5.1). Hence the equation of the line at infinity is $ax+by+cz=0$.

7.4. Let $\lambda=\nu=\mu'$. The M and μ of (7.3) now become M' and μ'. Note that the sides of $\triangle\mu'$ are isogonally conjugate to the sides of $\triangle\mu$.

7.5. Writing again a, b, c for f, g, h in (7.4), the point M' becomes the point K' of coordinates $(1/a, 1/b, 1/c)$, and μ' becomes the conic $(1/a, 1/b, 1/c)$. By (7.4), the tangents to this conic at A, B, C are isogonally conjugate to the tangents to the conic (a, b, c), which is the circumcircle (7.3). These latter are antiparallel to BC, CA, AB (2.1). Thus the tangents to the conic $(1/a, 1/b, 1/c)$ are parallel to BC, CA, AB. This conic is therefore Steiner's ellipse, and hence K' is the centroid G.

7.6. In (7.1), take $\lambda=\mu$ and $\nu=\mu'$. The point X now has coordinates $(h/g, f/h, g/f)$ and the hexagon of (7.1) consists of the sides of $\triangle\mu$ and of $\triangle\mu'$.

Let $\angle XAB = \alpha$, $\angle XBC = \beta$, $\angle XCA = \gamma$. Then the second and third of the trilinear coordinates of X are proportional to $\sin(A-\alpha)$, $\sin\alpha$ or

$$\sin(A-\alpha) : \sin\alpha = f^2 : gh$$

with symmetrical expressions for β and γ. Hence

$$\sin(A-\alpha) : \sin(B-\beta) : \sin(C-\gamma) = f^3 : g^3 : h^3,$$

and

$$\cot\alpha = \cot A + (f^2/gh \sin A).$$

Then $\cot\beta$ and $\cot\gamma$ are obtained by symmetry.

7.7. If in 7.6 we put $f=a$, $g=b$, $h=c$, then

$$\begin{aligned}\cot\alpha &= \cot A + (\sin A/\sin B \sin C)\\ &= \cot A + \cot B + \cot C.\end{aligned}$$

Hence by symmetry, $\cot\alpha = \cot\beta = \cot\gamma$, and hence $\alpha=\beta=\gamma=\omega$, where ω is the Brocard angle (3.0); the relations of 7.6 become

$$\cot\omega = \cot A + \cot B + \cot C$$

and

$$\sin(A-\omega) : \sin(B-\omega) : \sin(C-\omega) = a^3 : b^3 : c^3,$$

and the point X of (7.6) now becomes the point $(c/b, a/c, b/a)$, which are therefore the coordinates of the Brocard point Ω (3.0). The point E of the hexagon of (7.1) is now the point of intersection of the tangent to the circle at C and of the parallel to BC through A (7.5). Thus $\angle EBC = \omega$.

7.8. Interchanging μ and μ' in (7.6) we obtain the other Brocard point Ω', the isogonal conjugate of Ω (3.0).

7.9. Proceeding along these lines we could re-state the properties of the Brocard figures in a more general form. I can sketch here only the generalisation for the cosine circle (2.2).

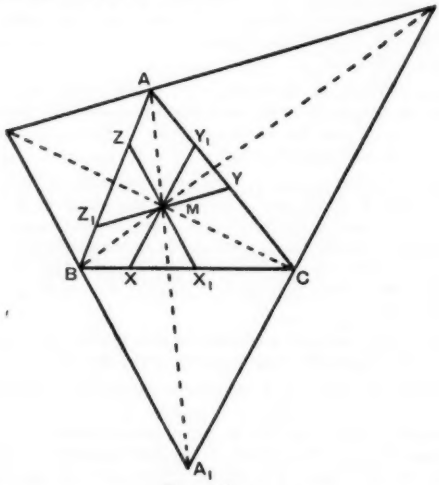


FIG. 17.

We first observe (Fig. 17) that AM and the tangent to μ at A form a harmonic pencil with AB, AC. This can be seen by considering the complete quadrilateral formed by the four lines AM, MC and the tangents at A and C,

and the transversal BM; or it follows analytically from the equations of AB, AC, namely $z=0$, $y=0$, and those of the tangent at A, $gz+hy=0$, and of AM, $gz-hy=0$.

Now let parallels through M to the sides of $\triangle\mu$ cut BC, CA, AB in X, X_1, Y, Y_1, Z, Z_1. From the harmonic range determined on Z_1Y by the above pencil, it follows that $Z_1M=MY$; hence, by symmetry, the parallels through M to the tangents to μ at A, B, C terminated by the sides of the triangle ABC are bisected at M. Thus the conic through five of these points has M as centre and must pass through the sixth point: X, X_1, Y, Y_1, Z, Z_1 lie on a conic whose centre is M.

7.10. Let F, G, H be the coordinates of the centre of μ, A_1 the point of intersection of the tangents to μ at B and C, that is, the point $(-f, g, h)$, and x and x_1 the actual x-coordinates of M and A_1. The coordinates of the centre of μ are given by

$$\frac{d\mu}{dF}\Big/a=\frac{d\mu}{dG}\Big/b=\frac{d\mu}{dH}\Big/c$$

(see, *e.g.* Smith, *Conic Sections*, p. 288, or Milne, *Homogeneous Coordinates*, p. 43).

Hence $F=f(-af+bg+ch)$, with similar expressions for G and H.

$$x=2\triangle f/(af+bg+ch),$$

and

$$x_1=-2\triangle f/(-af+bg+ch)$$
$$=-2\triangle f^2/F.$$

The triangles MXX_1 and A_1CB are similar (Fig. 17). Hence

$$XX_1=ax/x_1=-(af+bg+ch)\,aF/f.$$

By symmetry we obtain

$$XX_1 : YY_1 : ZZ_1=aF/f : bG/g : cH/h.$$

From the triangle BXZ_1 we have $BX=XZ_1 \,.\, a/b=YY_1 \,.\, a/b$, and by symmetry

$$BX_1 : XX_1 : X_1C=G/g : F/f : H/h.$$

7.11. Writing a, b, c in place of f, g, h, M becomes K (7.3). Therefore the symmedian lines (2.1) are bisected in K (7.9) and, since the centre of the circumcircle has coordinates $(\cos A, \cos B, \cos C)$ (1.5), the relations in 7.10 become

$$BX : XX_1 : X_1C=\cot B : \cot A : \cot C,$$

and

$$XX_1 : YY_1 : ZZ_1=\cos A : \cos B : \cos C,$$

the properties of the cosine circle (2.2).

The properties of the triplicate-ratio circle (3.3) can be developed on similar lines.

7.12. Calling the point M (f, g, h) and the line m (f, g, h) dual elements or duals, it follows from the duality principle that the duals of a pencil of lines through M are points on the line m, and conversely. If the arbitrary circumconic μ in (7.10) is chosen as the conic generated by the line at infinity in the general system of homogeneous coordinates, then the dual of M, that is, the line m, is the line at infinity. Thus a system of parallel lines, being a pencil whose common point is at infinity, has as its dual a set of points on the dual of a point at infinity; that is, the duals of a system of parallel lines are points on a line through M. In the trilinear system, M is K, the symmedian point (7.3), and in the areal system M is G, the centroid (7.5). Thus: the duals of a system of parallel lines are (*a*) on a line through the symmedian point in the trilinear system; (*b*) on a line through the centroid in the areal system.

8. *Examples.*

8.1. To prove Pascal's theorem by isogonal conjugates.

Let $AFBDCE$ be a hexagon, and let the opposite sides BD, AE meet in P,

DC, AF meet in Q, CE, FB meet in R. Take ABC as the triangle of reference, and let the coordinates of D, E, F be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). The equation of BD is then $xz_1 - x_1z = 0$, the equation of AE is $yz_2 - y_2z = 0$, and hence the coordinates of P are $(x_1/z_1, y_2/z_2, 1)$, and those of Q and R can be written down by symmetry. Now suppose that the six points A, B, C, D, E, F are on a conic, μ, say. The conjugate points D', E', F' have coordinates $(1/x_1, 1/y_1, 1/z_1)$, etc., (1.4), and they lie on the generating line m (5.0). The condition for this collinearity is

$$\begin{vmatrix} 1/x_1 & 1/y_1 & 1/z_1 \\ 1/x_2 & 1/y_2 & 1/z_2 \\ 1/x_3 & 1/y_3 & 1/z_3 \end{vmatrix} = 0$$

which can be written

$$\begin{vmatrix} 1 & x_1/y_1 & x_1/z_1 \\ y_2/x_2 & 1 & y_2/z_2 \\ z_3/x_3 & z_3/y_3 & 1 \end{vmatrix} = 0$$

where the columns are the coordinates of P, Q, R; hence P, Q, R are collinear.

8.2. To find the equation of the tangent to the conic $\mu \equiv fyz + gzx + hxy = 0$ at A.

In Fig. 8, the line m cuts BC at X. Hence $u \equiv gy + hz = 0$ is the equation of AX, and so $u' \equiv hy + gz = 0$ is the equation to AX', which is the tangent at A (5.7).

8.3. If m is a straight line (trilinear system), the conic μ and the circle intersect in A, B, C and D. To construct the point D (Fig. 18).

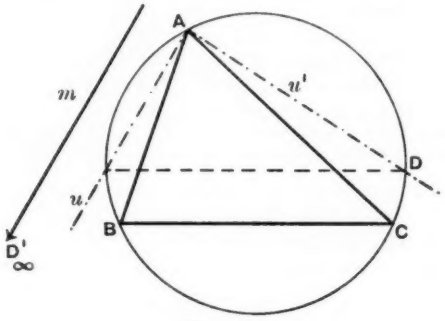


FIG. 18.

The corresponding point D' is at infinity (5.1). Draw a line u through A parallel to m; then u' is constructed as in (5.2) and meets the circle in the required point (1.3).

8.4. To prove that any rectangular hyperbola circumscribing a triangle passes through its orthocentre, and to find the condition that $fyz + gzx + hxy = 0$ should represent a rectangular hyperbola.

By (5.12), a rectangular hyperbola is generated by a line through the circumcentre O, whose coordinates are $(\cos A, \cos B, \cos C)$ (1.5). Hence O', the conjugate of O, is on the rectangular hyperbola (5.0); but O' is the orthocentre H (1.5). This proves the first result, and since the coordinates of H are $(\sec A, \sec B, \sec C)$ (1.5, 5.0) the required condition is

$$f \cos A + g \cos B + h \cos C = 0.$$

8.5. To construct the two parabolas which can be drawn through four concyclic points A, B, C, D (Fig. 19).

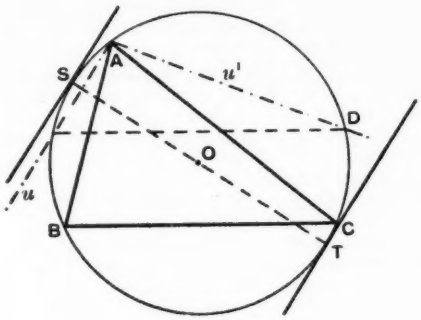


FIG. 19.

Take ABC as triangle of reference, and let u' be the line AD. Construct u (5.2). Then the two tangents to the circle which are parallel to u, touching the circle at S and T, generate the two parabolas required (5.10, 8.3). Now it is easily proved that the centroid of four concyclic points on a parabola lies on the axis. Hence if G_1 is the centroid of the four points A, B, C, D, the parallels through G_1 to the Simson lines of S and T are the axes of the two parabolas (5.13). The two tangents at A can be drawn (5.7). The focus and directrix for each curve can now be found by familiar methods; for instance, the tangent at A will meet the axis at T, and if F is the focus, it is well known that $TF = FA$.

8.6. In Fig. 20, $\angle BCD = \angle CBD = \angle ACE = \angle CAE = \angle BAF = \angle ABF = \alpha$. AD, BE, CF are then concurrent in a point X. To construct the locus of X for a variable α.

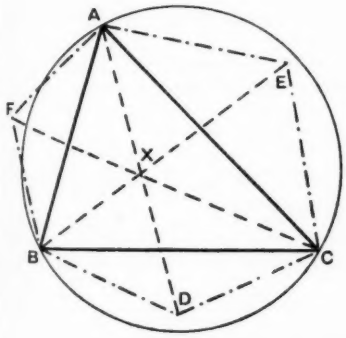


FIG. 20.

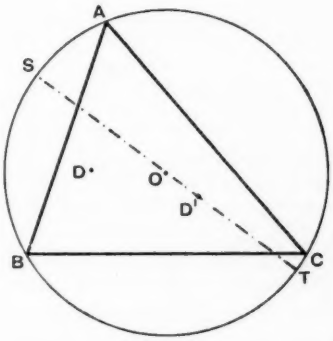


FIG. 21.

The locus is the circumconic passing through the orthocentre H and the centroid G (Sommerville, *Analytical Conics*, p. 166). But G' is the symmedian

point K (1.5). Hence the locus is the rectangular hyperbola generated by the line $H'G'$, that is, by the line OK (1.5, 2.2). Let OK cut the circle at S and T; the Simson lines of S and T are the asymptotes (5.13). Hence the asymptotes can be drawn in position. The familiar intercept property, that a chord PP_1 of a hyperbola meets the asymptotes in points QQ_1 such that $QP = P_1Q_1$, can now be used to plot the curve.

8.7. The nine-points circles of the four triangles determined by four points A, B, C, D are known to be concurrent in a point N (Forder, *Higher Course Geometry*, p. 233). To find N.

Construct the conjugate D' of D with respect to the triangle ABC (Fig. 21). Join OD', cutting the circle in S and T. Then the Simson lines of S and T intersect in the required point (5.13). This also proves that the centre locus of the family of rectangular hyperbolas circumscribing a triangle is the nine-points circle.

8.8. To construct Steiner's ellipse by isotomic conjugates.

In Fig. 22, the isotomic conjugates u', v' of parallel lines u, v through A, B respectively can be constructed by (6.2), and u', v' meet on Steiner's ellipse,

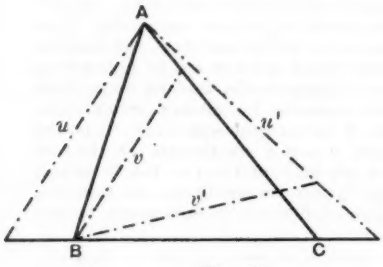


FIG. 22.

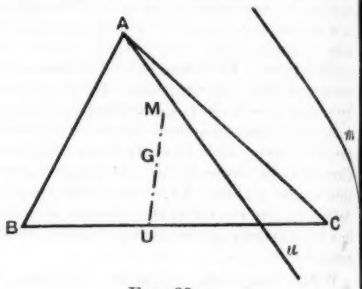


FIG. 23.

by (6.1). Hence we can obtain as many points as we please on Steiner's ellipse.

8.9. To find the radius of the cosine circle, from the generalisation in (7.9).

We first calculate ZM in Fig. 17, and obtain

$$ZM = 2bc(g^2 + h^2 - 2gh\cos A)^{\frac{1}{2}}/(af + bg + ch).$$

Now take f, g, h to be a, b, c, when M becomes the symmedian point K, and we have

$$ZK = 2abc/(a^2 + b^2 + c^2) = R\tan\omega.$$

(Compare (2.3) and (3.1)).

8.10. Given a line m, its dual M, and a line u through A parallel to m. To construct U, the dual of u, in (a) trilinears, (b) areals.

(a) Join M to the symmedian point K. Then MK cuts BC in the required dual (7.12).

(b) Join M to the centroid G. Then MG cuts BC in the required dual (7.12). This case (b) is shown in Fig. 23. A. S.

GLEANINGS FAR AND NEAR.

1536. The Constellations . . . carry 26 passengers on the Westward flight in addition to the mail. On the East-West flight the machines carry 35 passengers.—*The Observer*, July 14, 1946. [Per Mr. G. L. Parsons.]

SOLUTION OF DIFFERENTIAL EQUATIONS BY OPERATIONAL METHODS.

By B. M. Brown.

Introduction.

The method of solving linear differential equations with constant coefficients by using the operator D, which is given in elementary textbooks, was introduced in 1841 by Boole. In spite of the developments summarised below it remains the best method available for certain types of problem. As usually presented it is a tentative process, justified only by substitution in the original equation of the result produced. It is used only to obtain particular integrals, although it can be adapted to deal with complementary functions. This is not done in practice, however, since these can usually be written down directly.

Towards the end of the last century Heaviside solved ordinary and partial differential equations of electromagnetic theory by manipulating the operator d/dt, which he denoted by p, according to certain rules. This method was applied to problems with initial conditions, specified or implied, corresponding to a quiescent state for negative values of the independent variable t. Heaviside offered little justification of his admittedly ingenious work other than verification. This was left to other writers, amongst whom Bromwich made a considerable contribution. He placed Heaviside's work on a solid foundation by using a certain type of contour integral. Later Carson was led, by analogy with standard methods used by electrical engineers for solving circuit problems, to approach the problem by means of an integral equation. Although Carson uses operational language his method is not operational. In fact at this stage we are led to the idea that the integrals of Bromwich and Carson define a functional transformation and its inverse, in which the solution of a linear differential equation with constant coefficients is equivalent to the solution of an algebraic equation.

With a slight modification this transformation is identical with the Laplace transformation, which, as its name suggests, has been known for some time. It is closely connected with the Fourier transformation.

In recent years a number of books have been written which develop the subject from the first by means of the Laplace transformation and its inverse, without using either the language or the idea of an operator. In view of its directness and its generality this procedure has much to commend it, and is probably the most satisfactory of all for teaching purposes.

The operational approach has certain objections, chief of which is its lack of generality. Normally it is adequate for ordinary differential equations, but for the types of function that arise in partial differential equations it ceases to be applicable strictly, and has to be supplemented by generalised transformation. The latter is therefore inevitable in some form in a complete development of the subject.

It is however a pity to short circuit the operational idea completely. For one reason it helps to link the Laplace transformation through the Fourier transformation with other operational processes. Secondly, occasions arise when operational processes are particularly convenient, as, for example, in application of shift theorems.

In addition there is much literature in circulation at present in which operational methods are used. In this the logic is very often questionable; in fact the starting-point is usually the bald statement "Let d/dt be denoted by p". The enquiring reader very naturally asks how p with its applications differs from the more familiar operator D. One purpose of this paper is to

develop and compare the two operational methods step by step, so that their points of similarity and their differences can be seen clearly.

It was stated at the beginning that the "D" method of Boole and the "p" of Heaviside were tentative. The second purpose of this paper is to overcome this objection and build the operational processes on a logical foundation, so that it is possible to decide in advance what steps are or are not permissible. In the case of D this will help to explain certain paradoxes which appear from time to time (see for example the *Gazette*, XVI, No. 218 (May 1932), Note 1027, and XXXI, No. 294 (May, 1947), Note 1955). The method to be used for p is fundamentally the same as that used by Jeffreys in his tract *Operational Methods in Mathematical Physics*, being based ultimately on definite integrals. The details however are different.

THE OPERATOR D.

Notation. The independent variable will be denoted throughout by t, and the operator d/dt will be denoted by D.

$Q(\lambda)$ and $R(\lambda)$ will denote polynomials in λ of degrees q and r respectively, and $F(\lambda) = Q(\lambda)/R(\lambda)$.

We shall assume the elementary properties of the operator D, such as the distributive and associative laws for polynomials in D, and also the elementary properties of linear differential equations with constant coefficients, including the particular integral and complementary function.

Then if u and its first q derivatives exist $Q(D)\,u$ exists and is uniquely defined.

We define $\dfrac{1}{R(D)}\,u$ to denote the class of functions y which satisfy the differential equation

$$R(D)y = u.$$

The expression for y will contain r arbitrary constants. Note that when we state that two expressions of this type are equal we are expressing the equivalence of the two classes and not the equality of two particular members of the classes.

A very important particular case is $\{1/R(D)\}0$, which is of course the solution of $R(D)\,y = 0$. This is the sum of r terms of exponential type in general. Its properties include the following:

$$\frac{1}{R(D)}\,u = \frac{1}{R(D)}\,u + \frac{1}{R(D)}\,0,$$

$$\frac{1}{R_1(D)\,R_2(D)}\,0 = \frac{1}{R_1(D)}\,0 + \frac{1}{R_2(D)}\,0,$$

provided that $R_1(D)$ and $R_2(D)$ have no common factor.

We shall now develop the properties of $F(D)$ by means of a series of theorems.

Theorem 1. $$Q_1(D)\,\{Q_2(D)u\} = \{Q_1(D)Q_2(D)\}\,u.$$

This follows immediately from the properties of D.

Theorem 2. $$\frac{1}{R_1(D)}\left\{\frac{1}{R_2(D)}\,u\right\} = \frac{1}{R_1(D)R_2(D)}\,u.$$

For if we denote the left-hand side by y, $R_1(D)y = v$ where $R_2(D)v = u$, so that y is any function satisfying $R_1(D)R_2(D)y = u$.

Theorems 1 and 2 show that both the direct and inverse operators are commutative. We now consider cases in which they occur together.

Theorem 3. (*a*) $Q(D)\,\dfrac{1}{Q(D)}\,u = u,$

(*b*) $\dfrac{1}{Q(D)}\,Q(D)u = u + \dfrac{1}{Q(D)}\,0.$

(*a*) follows directly from definition. To prove (*b*) denote the left-hand side by y. Then $Q(D)y = Q(D)u$ so that $Q(D)\,(y-u) = 0$, and $y - u = \dfrac{1}{Q(D)}\,0$.

In particular $D\,\dfrac{1}{D}\,u = u$ and $\dfrac{1}{D}\,Du = u + A$ where A is an arbitrary constant.

Theorem 4. If $y_1 = \dfrac{1}{R(D)}\left\{\,Q(D)u\,\right\}$ and $y_2 = Q(D)\left\{\,\dfrac{1}{R(D)}\,u\,\right\}$, then $y_1 = y_2$, provided that $Q(D)$ and $R(D)$ are prime. If they have a common factor $S(D)$, then $y_1 = y_2 + \dfrac{1}{S(D)}\,0$.

Proof. It is obvious that y_1 is identical with the class y where $R(D)y = Q(D)u$. Also y_2 belongs to the class, for

$$R(D)y_2 = R(D)\,Q(D)\,\frac{1}{R(D)}\,u = Q(D)R(D)\,\frac{1}{R(D)}\,u = Q(D)u$$

by theorems 1 and 3 (*a*).

It follows that if y_1 and y_2 differ they must do so in the complementary function terms $\{1/R(D)\}0$. Now y_1 contains all the terms of $\{1/R(D)\}0$, and so does $\{1/R(D)\}u$; y_2 will do so unless the operation with $Q(D)$ causes some or all of the terms of $\{1/R(D)\}0$ to vanish. Let a typical term be $Ae^{\lambda t}$. Then $Q(D)Ae^{\lambda t} = Q(\lambda)Ae^{\lambda t}$. This will vanish if and only if $Q(\lambda) = 0$, that is if $Q(D)$ and $R(D)$ have a common factor $D - \lambda$. A similar proof is available for repeated factors of $R(D)$. Hence the theorem is proved.

The operator $F(D) = \dfrac{Q(D)}{R(D)}$ is so far undefined. We now define this by stipulating that the inverse operation shall be applied *after* the direct operation.

Definition. $$F(D)u = \frac{Q(D)}{R(D)}\,u = \frac{1}{R(D)}\,\{Q(D)u\}.$$

Theorem 4 shows that with this definition $F(D)u$ includes the class $Q(D)\{1/R(D)\}u$, and may be equivalent to it. We choose this definition because of its greater generality. We observe that $F(D)\,u$ contains all the terms represented by $\{1/R(D)\}0$.

We are now in a position to state rules for the algebraic manipulation of rational functions of D when solving differential equations or otherwise reducing expressions containing operators.

Rule 1. By virtue of theorems 1, 2 and 4 we may apply ordinary algebraic processes to $F(D)$, subject to the reservations of rules 2 and 3.

Rule 2. If a factor $S(D)$ is cancelled from the numerator and denominator the terms $\{1/S(D)\}0$ will disappear.

Usually (but not always) these terms will appear elsewhere, in which case their disappearance is not serious.

Rule 3. Numerator and denominator of an operator may not be multiplied by $S(D)$ unless subsequently the direct operation is performed *after* the inverse operation.

Theorem 4 shows that if this rule is not observed the spurious terms $\{1/S(D)\}0$ will be introduced. This is one source of the paradoxes referred to in the introduction.

Theorem 5. $$F(D)e^{\lambda t}u = e^{\lambda t}F(D+\lambda)u,$$

or $$F(D)u = e^{\lambda t}F(D+\lambda)e^{-\lambda t}u.$$

Proof. $$(D+\alpha)e^{\lambda t}u = e^{\lambda t}(D+\alpha+\lambda)u.$$

Also $$\frac{1}{D+\alpha}e^{\lambda t}u = e^{\lambda t}\frac{1}{D+\alpha+\lambda}u,$$

since if $$y = \frac{1}{D+\alpha}e^{\lambda t}u \text{ then } e^{\lambda t}u = (D+\alpha)y = e^{\lambda t}(D+\alpha+\lambda)e^{-\lambda t}y.$$

The theorem in the general case follows by factorising the numerator and denominator of $F(D)$ and by repeated application of these results. For example,

$$(D+\alpha)(D+\beta)e^{\lambda t}u = (D+\alpha)\{e^{\lambda t}(D+\beta+\lambda)u\} = e^{\lambda t}(D+\alpha+\lambda)(D+\beta+\lambda)u.$$

Theorem 6. $$\frac{1}{D-\alpha}u = e^{\alpha t}\int e^{-\alpha t}u\,dt = \int_0^t e^{\alpha(t-s)}u(s)\,ds,$$

For by theorem 5, $$\frac{1}{D-\alpha}u = e^{\alpha t}\frac{1}{D}e^{-\alpha t}u.$$

Solution of differential equations. The solution of the differential equation $R(D)y = u$ is equivalent to the reduction of $y = \{1/R(D)\}u$ to a simple form.

For a theoretical treatment several methods are available. Firstly we may split $R(D)$ into linear factors and apply theorem 6 repeatedly. Alternatively we may express $1/R(D)$ in partial fractions and apply theorem 6 to each term, provided there are no repeated factors.

In practice it is usually most convenient to evaluate the complementary function separately and use operational methods to obtain the particular integral. Terms belonging to $\{1/R(D)\}0$ in the latter are not usually written down.

Useful formulae. The following formulae are particularly useful for the above purpose. They can all be deduced from theorems 1 to 6 or verified easily.

$$\frac{1}{D}0 = A. \quad \frac{1}{D^n}0 = A_0 + A_1t + A_2t^2 + \ldots + A_{n-1}t^{n-1}.$$

$$\frac{1}{D+\lambda}0 = \frac{1}{D+\lambda}e^{-\lambda t}0 = e^{-\lambda t}\frac{1}{D}0 = Ae^{-\lambda t}.$$

$$\frac{1}{(D+\lambda)^n}0 = e^{-\lambda t}(A_0 + A_1t + \ldots + A_{n-1}t^{n-1}).$$

$$Q(D)1 = Q(0). \quad \frac{1}{D^n}1 = \frac{t^n}{n!}.\dagger$$

$$\frac{1}{R(D)}1 = \frac{1}{R(0)}\dagger \text{ provided } R(0)\neq 0.$$

If $$R(D) \equiv D^nR_n(D) \text{ where } R_n(0)\neq 0,$$

$$\frac{1}{R(D)}1 = \frac{1}{D^n}\frac{1}{R_n(0)} = \frac{t^n}{n!R_n(0)} = \frac{t^n}{R^{(n)}(0)}.\dagger$$

$$F(D)1 = F(0)\dagger \text{ provided } R(0)\neq 0.$$

† These expressions should include the expression $\{1/R(D)\}0$, or similar complementary function terms, for completeness. As stated above they are usually omitted.

$$F(D)e^{\lambda t} = e^{\lambda t}F(D+\lambda)\,1$$
$$= F(\lambda)e^{\lambda t} \dagger \text{ provided } R(\lambda) \neq 0.$$
$$Q(D^2)\sin \lambda t = Q(-\lambda^2)\sin \lambda t.$$
$$\frac{1}{R(D^2)}\sin \lambda t = \frac{1}{R(-\lambda^2)}\sin \lambda t \dagger \text{ provided } R(-\lambda^2) \neq 0.$$
$$\frac{1}{R(D)}\sin \lambda t = R(-D)\frac{1}{R(D)R(-D)}\sin \lambda t$$
$$= R(-D)\frac{1}{\phi(-\lambda^2)}\sin \lambda t$$
$$= \frac{1}{\phi(-\lambda^2)}R(-D)\sin \lambda t, \dagger$$

where $\phi(D^2)$ is written for $R(D)R(-D)$, which is clearly an even function of D. The last three formulae hold of course if $\sin \lambda t$ is replaced by $\cos \lambda t$ or $\sin(\lambda t + \epsilon)$.

It will be seen that the process adopted in the last formula does not violate rule 3. In practice the procedure used is slightly different, and is as shown in the following example.

$$\frac{1}{D^2+4D+1}\sin 2t = \frac{1}{-4+4D+1}\sin 2t$$
$$= \frac{4D+3}{(4D-3)(4D+3)}\sin 2t = \frac{4D+3}{16D^2-9}\sin 2t$$
$$= -\frac{1}{73}(4D+3)\sin 2t = -\frac{8}{73}\cos 2t - \frac{3}{73}\sin 2t.$$

This is justifiable inasmuch as it leads to the same result provided no other variations are introduced, but the partial substitution for D at the first step is not strictly permissible as it implies the introduction of $\{1/(4D-3)\}0$, which is definitely incorrect. It is also possible in this way to derive a false particular integral.

Expansion of the operator. We shall prove that the operator $F(D)$ is equivalent to its expansion in ascending powers of D, but not in general to its expansion in descending powers.

If we expand in ascending powers of D, the remainder after the term in D^n will be of the form $D^{n+1}S_n(D)/R(D)$, so that

$$\frac{Q(D)}{R(D)} \equiv \sum_{m=0}^{n} a_m D^m + D^{n+1}\frac{S_n(D)}{R(D)}.$$

This can be obtained in conformity with rules 1, 2 and 3 from the identity

$$\frac{Q(D)}{R(D)} \equiv \frac{\sum_{m=0}^{n} a_m D^m R(D) + D^{n+1}S_n(D)}{R(D)}$$

If however we expand in descending powers, the appropriate identity is

$$\frac{Q(D)}{R(D)} \equiv \sum_{m=0}^{n}\frac{b_m}{D^m} + \frac{1}{D^{n+1}}\frac{T_n(D)}{R(D)},$$

which is equivalent to

$$\frac{Q(D)}{R(D)} \equiv \frac{(b_0 D^{n+1} + b_1 D^n + \ldots + b_n D)R(D) + T_n(D)}{D^{n+1}R(D)}$$

† See previous footnote.

To obtain the right-hand side from the left we must multiply numerator and denominator by D^{n+1}, thus violating rule 3.

If we expand in ascending powers and operate on a polynomial, the series will terminate after a finite number of terms. In other cases the series, being equivalent to one obtained by repeated integration by parts, is often asymptotic.

Conclusion. The cardinal point of the preceding presentation of the operational theory is the inclusion, actual or implied, of complementary function terms at every stage. The reason for this is that it is impossible in theory to regard any one particular integral as more fundamental than the others, so that there is no single function that can be regarded as arising naturally as the result of inverse operation. Adherence to this rule provides a system of operators that can be treated algebraically, with the qualifications of theorems 3 and 4 and rules 1, 2 and 3.

In practice, complementary function terms are often omitted from intermediate working, and are only included in the final result. The only respect in which the particular integral obtained by this method differs from any other is in simplicity of form. This renders justification of operational processes difficult, and sometimes leads to different results when different methods of reduction are used.

THE OPERATOR p.

In the previous section we developed an operational calculus characterised by the fact that inverse operations were complete primitives, in particular the operator $1/D$ represented an indefinite integral. We discovered that with the exceptions implied by theorems 3 and 4 the operational algebra was commutative.

In the present section we shall introduce an operational calculus in which inverse operators correspond to definite integrals. We shall find that a completely commutative algebra results. This is offset, however, as we shall see later, by a restriction on the direct operator, corresponding to the restriction implied by the relation between definite and indefinite integrals.

Notation. The values of y, dy/dt, d^2y/dt^2, ... when $t=0$ will be denoted by $y_0, y_1, y_2, \dots$; $[y]_n=0$ implies the n equations $y_0=y_1=y_2=\dots=y_{n-1}=0$.

Definition. $\dfrac{1}{R(p)}\,u$ is the particular integral of the differential equation $R(D)y=u$ which vanishes when $t=0$, together with its first $r-1$ derivatives. That is, $y=\dfrac{1}{R(p)}\,u$ is the particular member of the class $\dfrac{1}{R(D)}\,u$ such that $[y]_r=0$.

In particular,
$$\frac{1}{p}\,u=\int_0^t u(s)\,ds.$$

Lemma. If $y=\dfrac{1}{R(p)}\,u$ and $[u]_k=0$ then $[y]_{r+k}=0$.

For if
$$R(D)=a_0+a_1D+\dots+a_rD^r$$
$$(a_0+a_1D+\dots+a_rD^r)y=u \qquad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$
and
$$y_0=y_1=\dots=y_{r-1}=0.$$

Putting $t=0$ in (1) gives $a_ry_r=u_0$. Since $u_0=0$ it follows that $y_r=0$. If we differentiate (1) successively and put $t=0$ we deduce in the same way $y_{r+1}=y_{r+2}=\dots=y_{r+k-1}=0$. Hence the lemma.

Theorem 7. $$\frac{1}{R_1(p)}\left\{\frac{1}{R_2(p)}\,u\right\}=\frac{1}{R_1(p)R_2(p)}\,u.$$

For if $$y=\frac{1}{R_1(p)}\left\{\frac{1}{R_2(p)}\,u\right\},$$

then $$R_1(D)y=v, \quad [y]_{r_1}=0,$$

where $$R_2(D)v=u, \quad [v]_{r_2}=0.$$

Eliminating v, $$R_1(D)R_2(D)y=u.$$

Also by the lemma $$[y]_{r_1+r_2}=0,$$

so that $$y=\frac{1}{R_1(p)R_2(p)}\,u.$$

At this stage we introduce the direct operator $R(p)$. To do this we remark that if the operator is to belong to a commutative algebra it must be such that it is cancelled by the inverse operator $1/R(p)$, already defined. $R(p)y$ can therefore represent nothing but $R(D)y$. However, we have laid down that application of the inverse operator gives an expression y, such that $[y]_r=0$. We therefore introduce $R(p)$ formally as follows.

Rule 4. If $R(D)y=u$ and $[y]_r=0$, we may replace $R(D)$ by $R(p)$ and solve the differential equation by treating the operator algebraically.

More generally, whenever we write $Q(p)u$ we denote by this $Q(D)u$, and in addition imply the conditions $[u]_q=0$. When these conditions are satisfied we say that the form $Q(p)u$ is *permissible*.

In particular, py stands for dy/dt, and implies $y_0=0$.

Theorem 8. $Q_1(p)\{Q_2(p)u\}=\{Q_1(p)Q_2(p)\}u$, whenever both expressions are permissible. The conditions for this are $[u]_{q_1+q_2}=0$. This follows directly from theorem 1 and the lemma.

Theorem 9. If $y_1=\dfrac{1}{R(p)}\{Q(p)u\}$ and $y_2=Q(p)\left\{\dfrac{1}{R(p)}\,u\right\}$ then $y_1=y_2$, provided both y_1 and y_2 are permissible.

Proof. The permissibility of $Q(p)u$ implies the conditions $[u]_q=0$. Then y_1 is the solution of $R(D)y_1=Q(D)u$ satisfying $[y_1]_r=0$.

Now $y_2=Q(D)\left\{\dfrac{1}{R(p)}\,u\right\}$ and $\left[\dfrac{1}{R(p)}\,u\right]_{q+r}=0$ by the lemma. It is easily verified that y_2 satisfies $R(D)y_2=Q(D)u$ and that $[y_2]_r=0$, so that y_1 and y_2 are identical.

We observe that y_1 is permissible if $[u]_q=0$, and that y_2 is permissible if $[u]_{q-r}=0$, or in all cases if $q\leqslant r$. The latter results follow from the lemma and are less restrictive on u than the first. We are therefore led to the following definition, being of more general application.

Definition. $\dfrac{Q(p)}{R(p)}\,u$ is defined as $Q(p)\left\{\dfrac{1}{R(p)}\,u\right\}$. Theorem 9 shows that if $\{Q(p)/R(p)\}u$ can be interpreted at all, we can always use the form y_2, whereas the form y_1 is only permissible provided $[u]_q=0$.

We are now in a position to state two rules of manipulation.

Rule 5. Any operator which is a function of p may be manipulated algebraically provided that the intermediate or final forms used are permissible.

Rule 6. $F(p)(u+v)=F(p)u+F(p)v$, provided both terms on the right-hand side are permissible, or become so as the result of a subsequent operation, $1/R_1(p)$.

Theorem 10. $$F(p)e^{\lambda t}u=e^{\lambda t}F(p+\lambda)u.$$

Proof. (*a*) If $F(p)=Q(p)$, theorem 5 shows that the two expressions are equal provided they are permissible. But if one is permissible so is the other, for it is easily verified that if $[u]_q=0$ then $[e^{\lambda t}u]_q=0$, and conversely.

(*b*) If $F(p)=1/R(p)$, theorem 5 shows that the two expressions (denoted by y_1 and y_2) belong to identical classes. Furthermore they satisfy identical conditions, for

$$R(D)y_1=e^{\lambda t}u \qquad [y_1]_r=0$$
$$R(D+\lambda)e^{-\lambda t}y_2=u \qquad [e^{-\lambda t}y_2]_r=0$$

The second condition is equivalent to $[y_2]_r=0$, which shows that $y_1=y_2$.

The proof in the general case follows by repeated application of (*a*) and (*b*) as in theorem 5.

Theorem 11. $$\frac{1}{p-\lambda}u=e^{\lambda t}\int_0^t e^{-\lambda s}u(s)\,ds.$$

Theorem 12. $$\frac{1}{(p-\lambda)^n}u=e^{\lambda t}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}e^{-\lambda s}u(s)\,ds.$$

Theorem 11 follows directly from theorem 6, and theorem 12 can be obtained by differentiating with respect to λ, or inductively.

Operations on the unit function.

$$\frac{1}{p}1=\int_0^t ds=t, \qquad \frac{1}{p^n}1=\frac{t^n}{n!}.$$

$$\frac{1}{p-\alpha}1=e^{\alpha t}\int_0^t e^{-\alpha s}ds=\frac{1}{\alpha}(e^{\alpha t}-1).$$

$$\frac{p}{p-\alpha}1=e^{\alpha t}, \qquad \frac{p}{(p-\alpha)^n}1=\frac{t^{n-1}}{(n-1)!}e^{\alpha t}.$$

The remaining standard operational forms can be deduced easily from these results.

Application to solution of differential equations.

1. To solve $R(D)y=u$ given $[y]_r=0$.

By rule 4 we may write $R(p)y=u$.

Operate with $1/R(p)$, $$y=\frac{1}{R(p)}u.$$

Express $\{1/R(p)\}u$ in partial fractions and evaluate by means of theorems 11 and 12.

2. If we can write $u=\Phi(p)1$ it is preferable to evaluate y as follows.

$$y=\frac{1}{R(p)}\Phi(p)1=\frac{p\Phi(p)}{pR(p)}1.$$

We then express $\Phi(p)/pR(p)$ in partial fractions and interpret by means of the standard forms for operations on the unit function.

3. To solve $R(D)\,y=u$ with general initial conditions.

Since y does not necessarily vanish when $t=0$ we are not entitled to replace dy/dt by py. We must use instead $p(y-y_0)$, where y_0 is the value of y when

$t=0$. As we shall subsequently operate with $1/R(p)$ we may separate the terms and write $dy/dt = py - py_0$, although the separate terms py and py_0 are not permissible (Rule 6).

Similarly $\quad d^2y/dt^2 = p^2y - p^2y_0 - py_1$, etc.

Substituting in the differential equation gives

$$R(p)y - a_1py_0 - a_2(p^2y_0 + py_1) - \ldots = u.$$

Operating with $1/R(p)$ leads to

$$y = \frac{1}{R(p)}\,u + \frac{a_1py_0 + a_2(p^2y_0 + py_1) + \ldots}{R(p)}\,1,$$

which can be reduced by methods 1 and 2 above.

Expansion of $F(p)$. In the above development it is not necessary to expand the operator $F(p)$ in a power series. If it is desired to do this for any purpose it is easily seen that in general we may use descending but not ascending powers. This is because the latter will ultimately lead to non-permissible forms, unless we are dealing with a function which vanishes initially with its derivatives of all orders, however large.

This contrasts strangely with functions of D which, the reader will recall, can be expanded in ascending but not descending powers.

It may be mentioned that the procedure adopted by Jeffreys is to define the operator $1/p$ (or Q) as the definite integral, and to define $F(p)$ as a power series, from which the normal properties may be deduced. B. M. B.

1537. (*a*) I think that the attitude of ordinary people . . . towards mathematics as a whole may be summed up as the same as their attitude to the police force; that is, knowing that it was invented to make life even more difficult, but unexpectedly glad of it every now and then.

(*b*) The interesting thing about them (tables) is that twelve twelves should have been fixed as the saturation point; one theory being that it represents a point exactly half way between what human nature ought to be able to carry in its head, and what it would be able to carry if it hadn't been bullied.

(*c*) To be fair to geometry as well as to those learning it, people do carry admiration of it with them for the rest of their lives; a rather nice feeling, when they see a triangle, that once they had it taped.—" Mathematics ", *Punch*, Sept. 3, 1941. [Per Rev. A. F. Mackenzie.]

1538. From his eighth year onwards, John [Stuart Mill] had not only to learn, but also to teach his younger brothers and sisters, who were numerous. Apart from the *Iliad* and *Odyssey*, Aeschylus, Sophocles, and Euripides, all the best Latin authors, a great deal of history, and a minute study of Roman Government, he had not time for much learning after instructing the younger members of the family; he seems to have mastered little else, before the age of twelve, except algebra and geometry, the differential calculus, and several other branches of higher mathematics.—Bertrand Russell, *Freedom and Organisation* (1934), p. 116.

1539. Optimism here, taking the bit in its teeth, contended that as there is in experience no such thing as a straight line, so there is practically no such thing as evil—a prophetic application, it would seem, of Riemannian geometry to morals; that what seems hopelessly bad will in the end be found to contain the good principle; and, quite illogically, that what seems to be good will actually prove to be so.—*The Cambridge History of American Literature*, vol. 2, p. 25. [Per Dr. A. Erdélyi.]

OPTICS AND ELECTRON OPTICS.

AN APPLICATION OF THE LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS.

By T. J. Hargest.

Introduction. Much progress in Science and Technology has been made from applications of the linear differential equation with constant coefficients. Extensions of the theory have made it necessary to consider cases when the coefficients are no longer constant, but are variable. The present inquiry is to investigate one of these applications.

The fundamental theory of optics owes a great debt to the general conceptions presented by Sir W. R. Hamilton between 1828 and 1837. He established the identity of the optical description of the path of a light-ray through refractive media and the mechanical description of the motion of a mass-point through a potential field by a comparison of Fermat's principle of least time with Lagrange's principle of least action. In 1897 J. J. Thomson developed electron mechanics and thirty years later the theory of electron optics was commenced.

Most of the work in optics quoted Hamilton's generalisations and contented itself with following light-rays refracted by a finite number of refracting surfaces where the refractive index changed suddenly. It is true that a theory for heterogeneous media was developed but this was generally regarded as mainly of academic interest. With the development of electron optics, however, where the refractive index changes continuously along the path of the electron, the subject of heterogeneous media has now come into its own and has now important practical applications.

With the usual notation, by Fermat's principle we have

$$\delta\int n\,ds = 0.$$

This will cover both ordinary optical theory and electron optical theory. For any curve joining two points on the path

$$x = f_1(z), \qquad y = f_2(z),$$

the length of the path

$$P = \int_{z_1}^{z_2} n \sqrt{\left\{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right\}}\, dz$$

where n is a function of x, y, z; that is,

$$P = \int_{z_1}^{z_2} \mu\, dz,$$

where μ is a function of x, y, z, x', y', where $x' = dx/dz$, $y' = dy/dz$.

Thus
$$\delta P = \int_{z_1}^{z_2} \delta\mu \,.\, dz$$

$$= \int_{z_1}^{z_2} \left(\frac{\partial\mu}{\partial x'}\,\delta x' + \frac{\partial\mu}{\partial y'}\,\delta y' + \frac{\partial\mu}{\partial x}\,\delta x + \frac{\partial\mu}{\partial y}\,\delta y + \frac{\partial\mu}{\partial z}\,\delta z\right) dz.$$

Now
$$\delta x' = \frac{d}{dz}(\delta x), \qquad \delta y' = \frac{d}{dz}(\delta y)$$

and hence

$$\delta P = \left[\frac{\partial\mu}{\partial x'}\,\delta x + \frac{\partial\mu}{\partial y'}\,\delta y\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} \left\{\left(\frac{d}{dz}\frac{\partial\mu}{\partial x'} - \frac{\partial\mu}{\partial x}\right)\delta x + \left(\frac{d}{dz}\frac{\partial\mu}{\partial y'} - \frac{\partial\mu}{\partial y}\right)\delta y\right\} dz.$$

The neighbouring curves will have common end-points, so that the first term vanishes. Thus we have

$$\frac{d}{dz}\left(\frac{\partial\mu}{\partial x'}\right)=\frac{\partial\mu}{\partial x},\quad \frac{d}{dz}\left(\frac{\partial\mu}{\partial y'}\right)=\frac{\partial\mu}{\partial y}.$$

Assuming symmetry of revolution implies that μ is a function of

$$x^2+y^2,\ x'^2+y'^2,\ xx'+yy',\ xy'-x'y\ ;$$

that is,

$$\mu=\mu_0+\mu_1(x^2+y^2)+\mu_2(x'^2+y'^2)+\mu_3(xx'+yy')+\mu_4(xy'-yx')+T,$$

where T consists of higher powers than the second, and μ_0, μ_1, μ_2, μ_3, μ_4 are functions of z. Thus

$$\frac{d}{dz}(2\mu_2x'+\mu_3x-\mu_4y)=2\mu_1x+\mu_3x'+\mu_4y'-T_{t_x},\quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{d}{dz}(2\mu_2y'+\mu_3y+\mu_4x)=2\mu_1y+\mu_3y'-\mu_4x'-T_{t_y},\quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where

$$T_{t_x}\equiv\frac{d}{dz}\left(\frac{\partial T}{\partial x'}\right)-\frac{\partial T}{\partial x},\qquad T_{t_y}\equiv\frac{d}{dz}\left(\frac{\partial T}{\partial y'}\right)-\frac{\partial T}{\partial y}.$$

Writing $\zeta=x+iy$, multiplying (1) and (2) by 1 and i respectively, and adding, we have

$$\frac{d}{dz}(2\mu_2\zeta'+\mu_3\zeta+\mu_4i\zeta)=2\mu_1\zeta+\mu_3\zeta'-i\mu_4\zeta'-(T_{t_x}+iT_{t_y})$$

or

$$2\mu_2\zeta''+\mu_3\zeta'+\mu_4i\zeta'+2\mu_2'\zeta'+\mu_3'\zeta+\mu_4'i\zeta-2\mu_1\zeta-\mu_3\zeta'+i\mu_4\zeta'=T_t,$$

where

$$T_t=-(T_{t_x}+iT_{t_y})\ ;$$

this reduces to

$$\zeta''+\zeta' f_1(z)+\zeta f_2(z)=T_1\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where

$$f_1(z)=(\mu_2'+i\mu_4)/\mu_2,$$
$$f_2(z)=(\mu_3'+i\mu_4'-2\mu_1)/2\mu_2,$$
$$T_1=T_t/2\mu_2.$$

The question of image formation is therefore reduced to equation (3) both for ordinary optics and for electron optics.

The usual substitution

$$\chi=\zeta\,.\,\exp\left\{\tfrac{1}{2}\int f_1(z)\,dz\right\}$$

gets rid of the second term and equation (3) becomes

$$\chi''+\chi\,.\,f_3(z)=T\,.\exp\left\{\tfrac{1}{2}\int f_1(z)\,dz\right\}$$

where

$$f_3(z)=f_2(z)-\tfrac{1}{2}f_1'(z)-\tfrac{1}{4}\{f_1(z)\}^2.$$

At this stage let us branch off into (1) ordinary optics, (2) electron optics.

1. Maxwell's well-known fish-eye problem makes $\mu_2=0$, etc. This alters the whole structure of the differential equation so that it cannot be used. In any case, the problem has been worked out in detail elsewhere, using Hamilton's equations. Also, Lenz has generalised the problem.

If the medium consists of parallel planes of equal refractive index, by taking the z-axis perpendicular to these planes μ is now a function of z only. The theory awaits practical cases in order to test the solution of the differential equation (3).

2. Electron optics. Axially symmetrical electrostatic field.

Laplace's equation in cylindrical coordinates, using axial symmetry about the z-axis, is

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{dz^2}=0.$$

Now $$E_r=-\partial V/\partial r, \qquad E_z=-\partial V/\partial z,$$

hence $$-\frac{\partial E_r}{\partial r}-\frac{1}{r}E_r-\frac{\partial E_z}{\partial z}=0$$

or $$\frac{\partial}{\partial r}(rE_r)=-r\frac{\partial E_z}{\partial z}.$$

Thus $$rE_r=-\tfrac{1}{2}r^2\frac{\partial E_z}{\partial z},$$

assuming that the field E_z remains constant over the integration ; that is,

$$E_r=-\tfrac{1}{2}r\frac{\partial E_z}{\partial z}=\tfrac{1}{2}r\frac{\partial^2 V}{\partial z^2}.$$

Assume $$V=a_0+a_2r^2+a_4r^4+\dots,$$

where a_0, a_2, a_4, ... are functions of z ; the power series clearly consists of even powers of r, from symmetry.

$$\frac{\partial V}{\partial r}=2a_2r+4a_4r^3+\dots,$$

$$\frac{1}{r}\frac{\partial V}{\partial r}=2a_2+4a_4r^2+6a_6r^4+8a_8r^6+\dots,$$

$$\frac{\partial^2 V}{\partial r^2}=2a_2+4.3a_4r^2+6.5a_6r^4+8.7a_8r^6+\dots,$$

$$\frac{\partial^2 V}{\partial z^2}=a_0''+a_2''r^2+a_4''r^4+\dots.$$

Thus $$\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}=a_0''+4a_2+(a_2''+4^2\,.\,a_4)r^2$$
$$+(a_4''+6^2\,.\,a_6)\,r^4+(a_6''+8^2\,.\,a_8)\,r^6+\dots,$$

whence
$$a_2=-a_0''/2^2,$$
$$a_4=-a_2''/4^2=a_0^{\mathrm{iv}}/2^2\,.\,4^2,$$
$$a_6=-a_4''/6^2=-a_0^{\mathrm{vi}}/2^2\,.\,4^2\,.\,6^2,$$

and so on. Thus writing $a_0=f(z)$,

$$V=f(z)-\frac{r^2}{2^2}f''(z)+\frac{r^4}{2^2\,.\,4^2}\,.\,f^{\mathrm{iv}}(z)-\frac{r^6}{2^2\,.\,4^2\,.\,6^2}f^{\mathrm{vi}}(z)+\dots,$$

$$E_r=-\frac{\partial V}{\partial r}=\frac{r}{2}f''(z)-\frac{r^3}{2^2\,.\,4}f^{\mathrm{iv}}(z)+\frac{r^5}{2^2\,.\,4^2\,.\,6}f^{\mathrm{vi}}(z)-\dots,$$

The radial force on the electron will be given by

$$m\ddot{r}=-eE_r=-\frac{er}{2}\frac{\partial^2 V}{\partial z^2}.$$

Now
$$\ddot{r}=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d}{dt}\left(\frac{dr}{dz}\,\dot{z}\right)$$
$$=\dot{z}\frac{d}{dz}\left(\frac{dr}{dz}\,\dot{z}\right)=\dot{z}^2\frac{d^2r}{dz^2}+\dot{z}\frac{dr}{dz}\frac{dz}{dz}.$$

From the energy equation, $z = \sqrt{(2eV/m)}$. Hence substituting this value, we have

$$\ddot{r} = \frac{2eV}{m}\frac{d^2r}{dz^2} + \frac{e}{m}\frac{dr}{dz}\frac{dV}{dz}$$

or
$$\frac{d^2r}{dz^2} + \frac{V'}{2V}\frac{dr}{dz} + \frac{V''r}{4V} = 0.$$

Comparing with (3),

the real part of $f_1(z) = V'/2V$,
the real part of $f_2(z) = V''/4V$.

If we write $u = r \exp\left(\int \frac{V'dz}{4V}\right) = rV^{1/4}$, then

$$\frac{d^2u}{dz^2} + \frac{3}{16}\frac{V'^2}{V^2}u = 0. \qquad \text{.....(4)}$$

(*a*) If $V'/V = 4/(1+z^2)$, this equation becomes

$$\frac{d^2u}{dz^2} + \frac{3u}{(1+z^2)^2} = 0,$$

of which a solution is readily seen to be

$$u = z/\sqrt{(1+z^2)},$$

so that the electron path is

$$r = zV^{-1/4}/\sqrt{(1+z^2)}.$$

(*b*) If $V'/V = ae^{-z}$, then

$$u = J_0(\sqrt{\tfrac{3}{16}}\, ae^{-z}).$$

For, $J_0(x)$ satisfies the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0;$$

if we make the change of variable $4x = a\sqrt{3} \cdot e^{-z}$, this equation becomes

$$\frac{d^2y}{dz^2} + \frac{3a^2e^{-2z}}{16}y = 0,$$

which is equivalent to (4) if $V'/V = ae^{-z}$. Hence in this case the electron path is

$$r = V^{-1/4} J_0(\tfrac{1}{4}\sqrt{3} \cdot ae^{-z}).$$

Axially symmetric magnetic field.

The vector potential A now satisfies the equations

$$-\operatorname{curl}\operatorname{curl} A = \frac{\partial H_z}{\partial r} - \frac{\partial H_r}{\partial z} = 0;$$

now
$$H_r = -\frac{\partial A}{\partial z}, \quad H_z = \frac{1}{r}\frac{\partial}{\partial r}(rA).$$

Hence
$$-\frac{1}{r^2}\frac{\partial}{\partial r}(rA) + \frac{1}{r}\frac{\partial^2}{\partial r^2}(rA) + \frac{\partial^2 A}{\partial z^2} = 0,$$

or
$$-\frac{A}{r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} + \frac{\partial^2 A}{\partial z^2} = 0.$$

On the axis, A is always zero, so let us assume that

$$A = a_1 r + a_3 r^3 + a_5 r^5 + \ldots + a_{2n+1} r^{2n+1} + \ldots.$$

Then
$$\frac{\partial A}{\partial r} = a_1 + 3a_3 r^2 + 5a_5 r^4 + \ldots + (2n+1)\, a_{2n+1} r^{2n} + \ldots,$$

$$\frac{\partial^2 A}{\partial r^2} = 3 \cdot 2a_3 r + 5 \cdot 4a_5 r^3 + \ldots + 2n(2n+1)\, a_{2n+1} r^{2n-1} + \ldots,$$

$$\frac{1}{r}\frac{\partial A}{\partial r} = \frac{a_1}{r} + 3a_3 r + 5a_5 r^3 + \ldots + (2n+1) a_{2n+1} r^{2n-1} + \ldots,$$

$$-\frac{A}{r^2} = -\frac{a_1}{r} - a_3 r - a_5 r^3 - \ldots - a_{2n+1} r^{2n-1} - \ldots,$$

$$\frac{\partial^2 A}{\partial z^2} = a_1'' r + a_3'' r^3 + a_5'' r^5 + \ldots + a_{2n+1}'' r^{2n+1} + \ldots,$$

whence

$$a_1'' = -8a_3,$$
$$a_3'' = -24a_5,$$
$$\ldots\ldots\ldots\ldots$$
$$a''_{2n-1} = -\{(2n+1)^2 - 1\} a_{2n+1}$$
$$= -2n(2n+2) a_{2n+1},$$

so that

$$A = a_1 r - \frac{a_1''}{2 \cdot 4} r^3 + \frac{a_1^{\mathrm{IV}}}{2 \cdot 4 \cdot 4 \cdot 6} r^5 \ldots + (-)^n \frac{a_1^{2n} r^{2n+1}}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots 2n(2n+2)} + \ldots.$$

Now

$$H_z = \frac{1}{r}\frac{\partial}{\partial r}(rA)$$

and $\frac{\partial}{\partial r}(rA) = a_1 \cdot 2r$, to the first term, so that when $r = 0$,

$$\frac{1}{r}\frac{\partial}{\partial r}(rA) = 2a_1 = H_{0z}$$

or

$$a_1 = \tfrac{1}{2} H_{0z},$$

$$A = \tfrac{1}{2} H_{0z} r - \tfrac{1}{2} r^3 \frac{H_{0z}''}{2 \cdot 4} + \tfrac{1}{2} r^5 \frac{H_{0z}^{\mathrm{IV}}}{2 \cdot 4 \cdot 4 \cdot 6} \ldots \tfrac{1}{2}(-)^n r^{2n+1} \times \frac{H_{0z}^{2n}}{2 \cdot 4 \cdot 4 \cdot 6 \ldots (2n)(2n+2)} + \ldots.$$

We have already used the fact that

$$\frac{\partial H_r}{\partial r} + \frac{H_r}{r} = -\frac{\partial H_z}{\partial z}$$

that is, $H_r = -\frac{r}{2}\frac{\partial H_z}{\partial z}$, approximately.

In the magnetic field we have both radial and angular acceleration. For the tangential force, with the usual notation,

$$mr\ddot{\phi} = e\dot{r} H_z - e\dot{z} H_r,$$

and, neglecting $\dot{r}$ in comparison with $\dot{z}$,

$$\ddot{\phi} = -\frac{e}{rm} H_r \dot{z} = \frac{e}{2m}\frac{\partial H_z}{\partial z} \cdot \dot{z},$$

whence

$$\dot{\phi} = eH_z/2m,$$

since

$$\dot{\phi} = 0 \text{ when } H_z = 0.$$

Radially,

$$m(\ddot{r} - r\dot{\phi}^2) = -er\dot{\phi} H_z + e\dot{z} H_\phi.$$

Because of axial symmetry, $H_\phi = 0$,

and
$$\ddot{r} = r\dot{\phi}^2 - (er/m)\,\dot{\phi}\,H_z$$
$$= -e^2 r\,H_z^2/4m^2.$$

As before,
$$\frac{2eV}{m}\frac{d^2r}{dz^2} + \frac{e}{m}\frac{dr}{dz}\frac{dV}{dz} = -\frac{e^2r}{4m^2}H_z^2,$$

that is,
$$\frac{2eV}{m}\frac{d^2r}{dz^2} + \frac{e}{m}\frac{dV}{dr}\left(\frac{dr}{dz}\right)^2 = -\frac{e^2r}{4m^2}H_z^2,$$

and neglecting the second term since it contains the square of dr/dz, we have
$$\frac{d^2r}{dz^2} = -\frac{er}{8mV}H_z^2,$$

where H_z is usually taken as $H_0/\{1+(z/a)^2\}^{\mu}$.

Comparing with (3),

the imaginary part of $f_1(z)$ is $iH_z\sqrt{(e/2mV)}$

the imaginary part of $f_2(z)$ is $\frac{1}{2}iH_z'\sqrt{(e/2mV)}$,

and
$$f_2(z) = f_2(z) - \tfrac{1}{2}f_1'(z) - \tfrac{1}{4}\{f_1(z)\}^2$$
$$= \tfrac{1}{2}H_z' \cdot \sqrt{(e/2mV)} - \tfrac{1}{2}H_z'\sqrt{(e/2mV)} - \tfrac{1}{4}H_z^2 \cdot (e/2mV)$$
$$= -eH_z^2/8mV.$$

Thus the differential equation (3) becomes
$$\zeta'' + \zeta' f_1(z) + \zeta f_2(z) = T_1,$$

where in the electromagnetic instrument

$$f_1(z) = V'/2V \quad + \quad iH_z\sqrt{(e/2mV)}$$
$$f_2(z) = V''/4V \quad + \quad \tfrac{1}{2}iH_z' \cdot \sqrt{(e/2mV)}$$
$$f_3(z) = \tfrac{3}{16}(V'/V)^2 \quad + \quad eH_z^2/8mV.$$

$f_3(z)$ is always real, giving rise to a real image if we are allowed to increase the strength of the field sufficiently. The term T_1 on the right deals with the aberrations, and if these are neglected $T_1 = 0$ and the equation
$$\zeta'' + f_1(z)\,\zeta' + f_2(z)\,\zeta = 0$$

becomes
$$\chi'' + f_3(z)\chi = 0.$$

Taking the electrostatic system, that is, the real coefficients,
$$\frac{d^2r}{dz^2} + \frac{V'}{2V}\frac{dr}{dz} + \frac{V''}{4V}r = \frac{r^3}{2V}\cdot\frac{V^{\mathrm{IV}}}{2^2\cdot 4} - \frac{r^5}{2V}\cdot\frac{V^{\mathrm{VI}}}{2^2\cdot 4^2\cdot 6} + \frac{r^7}{2V}\cdot\frac{V^{\mathrm{VIII}}}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \ldots$$

will give an approximate value of the real part of T_1 suitable for commencing work on the evaluation of the aberrations.

In the case of the paraxial equation for the magnetic field Goddard and Klemperer have calculated the path of a 500 ev electron with $H_0 = 16{\cdot}5$ oersteds and an initial slope of path given by $\tan\theta = 0{\cdot}058$.

z	r	z	r
−12	1·166	0	1·7268
−10	1·282	2	1·6751
− 8	1·3978	4	1·5666
− 6	1·5119	6	1·4342
− 4	1·619	8	1·2948
− 2	1·7014	10	1·1546

In the derivation of the differential equation
$$\frac{d^2r}{dz^2} = -\frac{er}{8mV}H_z^2$$

the assumption was made that dr/dz was small and also in the expression $H_r = -\frac{1}{2}r\,\partial H_z/\partial z$ higher powers of r were neglected. Allowing for these assumptions, we may now proceed to obtain the imaginary part of T_1 as in the electrostatic case given above in extra powers of r. This should lead to expressions for the aberrations for the complete electromagnetic instrument, thus checking what experimental values are available. The theory in this field of work has not yet caught up with practice. Even in ordinary optics, practice has not been used to check the Hamiltonian generalised theory and Hamilton himself seems to have designed only one lens. Hamilton's T-function was a wonderful invention but it is pretty certain that few designers ever use it. In the same way designers of electromagnetic instruments prefer to work by experiment, observation and instinct rather than attempt to master complicated theories, where one might get lost trying to see the wood for the trees.

T. J. H.

1540. Let us represent our human individuals, whose social relations we are trying to grasp, by any number of points on a plane ; and from each one of these points, as an apex, let us produce a cone to represent each individual's field of action. Some of these cones may expand rapidly, at an obtuse angle; others may expand gradually, at angles that are more acute ; and the central spine or axis of each cone may take off at any angle from the common plane on which all the apices lie. Our only postulate is that the cones shall all be produced into the third dimension on one and the same side of our original plane (which offers, of course, two alternative sides on which our three-dimensional figures may be constructed). Let us now decree that every one of these cones shall be produced, on the (conventionally determined) productive side of our plane, from its apex on and on into Infinity. This decree ensures mathematically that, notwithstanding the diversity of the respective angles at which our cones take off from our plane, the ultimate cross-section of every cone will coincide in area with the ultimate cross-section of every other cone ; for, on a plane that lies at an infinite distance from our original plane of departure, all our cones alike will project themselves in cross-sections with infinite and therefore co-extensive areas. At the same time, we shall find that the sections of a certain number—though only a certain number—of our cones do coincide in area at each and every distance, short of infinity, from the productive face of our original plane, at which we choose to describe another plane parallel to it. For, as our cones which we are producing from our original plane traverse any one of these other planes, at some finite distance from the first, on their way towards Infinity, they will each leave their mark on the second plane in the shape of a conic section ; and while these conic sections will naturally differ from one another in location and size and form according to the point on the original plane from which each particular cone has taken off, and according to the angles that govern its production, it will be found, as a matter of fact, that the conic sections imprinted on the second plane by a certain number of different cones that have taken off from the first plane at different points and at different angles will actually coincide with one another—coinciding, in this case, at a finite and not at an infinite distance from their respective apices, and in a finite and not in an infinite area.—A. J. Toynbee, *A Study of History* (Oxford, 1934), ii, pp. 227–8.

1541. NECRO-MATHEMATICS.

A skeleton is defined as a function $f(r)$ defined only for rational values of r. . . . They also give inequalities for the upper and lower limits of $B(f, x)$ for any bounded skeleton.—From a review in *Mathematical Reviews*, VII, p. 440 (October, 1946). [Per Mr. R. A. Fairthorne.]

ON THE NODES OF A RATIONAL PLANE CURVE.

By Harold Simpson.

Consider a rational (unicursal) plane curve of degree n given by

$$x : y : z = f(t) : \phi(t) : \psi(t),$$

where

$$f(t) \equiv a_0 + a_1 t + a_2 t^2 + \ \dots\ + a_n t^n,$$
$$\phi(t) \equiv b_0 + b_1 t + b_2 t^2 + \ \dots\ + b_n t^n,$$
$$\psi(t) \equiv c_0 + c_1 t + c_2 t^2 + \ \dots\ + c_n t^n.$$

The cusps are given by

$$f'(t)/f(t) = \phi'(t)/\phi(t) = \psi'(t)/\psi(t).$$

The values α and β of t at a node are given by

$$f(\alpha)/f(\beta) = \phi(\alpha)/\phi(\beta) = \psi(\alpha)/\psi(\beta).$$

The usual method of finding α and β is to solve the equations

$$\{f(\alpha)\,\psi(\beta) - f(\beta)\,\psi(\alpha)\}/(\alpha - \beta) = 0, \quad \{\phi(\alpha)\,\psi(\beta) - \phi(\beta)\,\psi(\alpha)\}/(\alpha - \beta) = 0.$$

This is often effective in simple cases; but it suffers from lack of symmetry and from the fact that the solution includes values of α and β which do not give the parameters of nodes, but are obtained from $\psi(\alpha) = \psi(\beta) = 0$.

The following method preserves symmetry. Let $\alpha + \beta \equiv u$, $\alpha\beta \equiv v$,

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots\dots & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ b_0 & b_1 & b_2 & b_3 & \dots\dots & b_{n-3} & b_{n-2} & b_{n-1} & b_n \\ c_0 & c_1 & c_2 & c_3 & \dots\dots & c_{n-3} & c_{n-2} & c_{n-1} & c_n \\ v & -u & 1 & 0 & \dots\dots & 0 & 0 & 0 & 0 \\ 0 & v & -u & 1 & \dots\dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots\dots & v & -u & 1 & 0 \end{vmatrix},$$

$$D \equiv \begin{vmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \dots\dots & \alpha^n \\ 1 & \beta & \beta^2 & \beta^3 & \dots\dots & \beta^n \\ 0 & 0 & 1 & 0 & \dots\dots & 0 \\ 0 & 0 & 0 & 1 & \dots\dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots\dots & 1 \end{vmatrix}.$$

Forming the product $\Delta \times D$ by the ordinary rule for multiplying determinants, we see that $\Delta = 0$. Changing the final $v, -u, 1, 0$ of the last row of Δ into $0, v, -u, 1$, we get a determinant Δ_1 which is also zero; and so are the determinants $\Delta_2, \Delta_3, \dots \Delta_{n-2}$ obtained by applying a similar process to the last two, three, $\dots$, $n-2$ rows of Δ. If we think of u and v as Cartesian coordinates, we get their values as the abscissa and ordinate of any one of the $\frac{1}{2}(n-1)(n-2)$ points through which all the $n-1$ curves of degree $n-2$ $\Delta = 0$, $\Delta_1 = 0, \dots, \Delta_{n-2} = 0$ pass.

In these statements a triple point at which $t = \alpha, \beta, \gamma$ is counted as three double points for which $t = \beta$ and γ, $t = \gamma$ and α, $t = \alpha$ and β; and so for quadruple points, etc. Also a cusp is counted as a node for which $u^2 = 4v$; as may be seen on replacing the second row of the determinant D by $0, 1, 2\alpha, 3\alpha^2, \dots, n\alpha^{n-1}$.

When we have obtained the values of u, v at a double point, we have

$$x : y : z = \{f(\alpha)+f(\beta)\} : \{\phi(\alpha)+\phi(\beta)\} : \{\psi(\alpha)+\psi(\beta)\}$$
$$= \{f(\alpha)-f(\beta)\}/(\alpha-\beta) : \ldots : \ldots,$$

for the coordinates of the double point, since $f(\alpha)+f(\beta)$, etc., are at once expressible in terms of u and v.

When $n=3$ we get (u, v) as the intersection of the "lines" $\Delta=0$, $\Delta_1=0$, where

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ v & -u & 1 & 0 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ 0 & v & -u & 1 \end{vmatrix}.$$

When $n=4$ we get (u, v) as one of the three points A, B, C common to the three "conics" $\Delta=0$, $\Delta_1=0$, $\Delta_2=0$, where

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ v & -u & 1 & 0 & 0 \\ 0 & v & -u & 1 & 0 \end{vmatrix}, \quad \Delta_1 \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ v & -u & 1 & 0 & 0 \\ 0 & 0 & v & -u & 1 \end{vmatrix},$$

$$\Delta_2 \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ 0 & v & -u & 1 & 0 \\ 0 & 0 & v & -u & 1 \end{vmatrix}.$$

Suppose that the units in Δ, Δ_1, Δ_2 are replaced by w and that the Jacobian $\partial(\Delta, \Delta_1, \Delta_2)/\partial(u, v, w)$ is formed. Let J be the result of replacing w by 1 in this Jacobian. Then $J=0$ is the equation of the lines BC, CA, AB taken together. By finding the asymptotes of $J=0$ in the usual manner, we get the lines BC, CA, AB separately; and thence the points A, B, C may be obtained.

For example, the quartic curve with equations

$$x : y : z = (2t+t^3) : (1-4t+9t^2) : (1+3t+2t^4)$$

has its double points given by

$$u^2-4uv+9v^2+2u-9v=0,$$
$$2u^2-7uv-4v^2+3u+6v-2=0,$$
$$2uv+v^2-2u+3v-4=0,$$

and the Jacobian curve is

$$(v-1)(u-2v+2)(u-4v+2)=0.$$

Hence the double points are obtained from

$$u-2v+2=u-4v+2=0,$$
$$u-4v+2=v-1=0,$$
$$v-1=u-2v+2=0,$$

whence $u=-2$, $v=0$ leading to α and $\beta=0$ and -2, giving a crunode $(0, 1, 1)$ of the quartic; $u=2$, $v=1$ leading to α and β both equal to 1, giving a cusp $(1, 2, 2)$; $u=0$, $v=1$ leading to α and β equal to $\pm i$, giving an acnode $(1, 4, 0)$.

If $n>4$, the task of finding the nodes is more laborious, but we can find an equation of degree $\frac{1}{2}(n-1)(n-2)$ for u (or v) as the abscissa of a point common to $\Delta=0$, $\Delta_1=0$, ... , $\Delta_{n-2}=0$ by routine processes of elimination. As a simple illustration we may take the quintic

$$x : y : z = 1 : (t+t^4) : t^5,$$

for which the equations $\Delta=0$, etc. are

$$u^3-2uv+1=0, \qquad u^2v-v^2+u=0,$$

$$uv^2+u^2-v=0, \qquad u^3+v^3-2uv=0.$$

These four equations have in common the six solutions of

$$u^6+4u^3-1=0, \qquad 2uv=u^3+1,$$

whence

$$u=-\tfrac{1}{2}(1\pm\sqrt{5}), \quad v=1\,; \quad u=-\tfrac{1}{2}(1\pm\sqrt{5})\,\omega, \quad v=\omega^2\,;$$

$$u=-\tfrac{1}{2}(1\pm\sqrt{5})\,\omega^2, \quad v=\omega,$$

where $\omega^2+\omega+1=0$. The corresponding nodes of the quintic are

$$(2,\ \alpha+\alpha^4+\beta+\beta^4,\ \alpha^5+\beta^5),$$

or

$$(2,\ u^4-4u^2v+2v^2+u,\ u^5-5u^3v+5uv^2)\,;$$

that is

$$\{2,\ (\mp\sqrt{5}-1),\ 2\}, \quad \{2,\ (\mp\sqrt{5}-1)\,\omega, \quad 2\omega^2\}, \quad \{2,\ (\mp\sqrt{5}-1)\,\omega^2,\ 2\omega\}.$$

H. S.

CORRESPONDENCE.

"ISOSCELES."

To the Editor of the *Mathematical Gazette*.

SIR,—Mr. Williamson's spells may bind his boys, but is it not a pity to divorce mathematics entirely from the still somewhat cultural classics?

I suggest that the boy who can be spell-bound by *SOS* and *ELE* can be taught to spell by correlating the *iso* of *isosceles* with the several "*iso*"-s he comes across in his geography, and that he can as readily realise that *sceles* is the basic part of skele-ton. In this way an interest may (perhaps) be directed to Greek, which provides him with most of his geographical and other iso tags. If on his first introduction to isosceles, it be (mis-) pronounced "iso-skeles", his orthography may be helped, and he will later be able to adopt the more orthodox pronunciation and (again perhaps) remember the spelling, with an unconscious widening of his non-mathematical knowledge.

I have used "bind" in the sense in which Mr. Williamson does, not with the connotation of R.A.F. slang! Yours, etc., I. FITZROY JONES.

COMPLEX NUMBER PHRASEOLOGY.

To the Editor of the *Mathematical Gazette*.

SIR,—The use of the words "real" and "unreal" in connection with numbers has long been felt to be unsatisfactory, particularly by such leading textbook writers as Mr. C. V. Durell. And yet the former word is a most natural word at the stage when it is usually first needed, in the theory of quadratics; this is particularly apparent in the graphical treatment of the subject.

Could the later objections be overcome by using the word *irreal* or *super-real*? Or would lovers of the English language object?

Yours, etc., T. G. C. WARD.

MATHEMATICAL NOTES.

1966. *On the solution of a triangle given two sides and the included angle.*

There is no necessity to wait until half-angle formulae are known to solve the triangle, if the following method is used.

Let a, B and c be the elements given, and let $a > c$.

Using projections of c on and perpendicular to a, we have

$$\tan C = \frac{c \sin B}{a - c \cos B}.$$

(This is true in all cases, whether B is obtuse or acute, but with beginners it would be better to consider separate cases.)

Now, *dividing numerator and denominator* by c, we get

$$\tan C = \frac{\sin B}{(a/c) - \cos B}.$$

This greatly simplifies the work, as $\sin B$ and $\cos B$ can immediately be found from the tables, while a/c is easily calculated and, being greater than unity, will not involve negative characteristics if logarithms are used. Having found C, A follows immediately and b can be found from the sine rule.

H. N. HASKELL.

1967. *Note on continuity.*

In a recent note on substitution in the definite integral I proved the theorem that if $g(x)$ and $f(g(x))$ are continuous in (a, b), then $f(t)$ is continuous in the interval bounded by the lower and upper bounds of $g(x)$ in (a, b); the proof given there is indirect and proceeds by showing that a discontinuity in $f(t)$ implies a discontinuity in $f(g(x))$. The following simple direct proof has since occurred to me.

Choose any α, β such that $a \leqslant \alpha < \beta \leqslant b$. Since $f(g(x))$ is continuous in (α, β) we can divide (α, β) into $n = n(k)$ parts by the points $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots, \alpha(n) = \beta$, such that

$$|f(g(X)) - f(g(x))| < 1/k$$

for any two x, X in the same part.

We may suppose α, β chosen so that $g(\alpha) \neq g(\beta)$, for this is possible unless $g(x)$ is constant, and if $g(x)$ is constant there is nothing to prove; in fact, we may without loss of generality suppose that $g(\alpha) < g(\beta)$.

Since $g(\alpha(n)) > g(\alpha(0))$ we can determine a *least* integer r_1 such that

$$g(\alpha(n)) \geqslant g(\alpha(r_1)) > g(\alpha(0)),$$

a least r_2, greater than r_1, such that $g(\alpha(n)) \geqslant g(\alpha(r_2)) > g(\alpha(r_1))$, and so on, up to $r_m = n$.

If s is the first integer, $r_{p-1} < s \leqslant r_p$, $p \leqslant m$, such that $g(\alpha(s)) > g(\alpha(r_{p-1}))$ then each interval $g(\alpha(r_{p-1}), g(\alpha(r_p))$ is contained in the interval $g(\alpha(s-1)), g(\alpha(s))$, *i.e.* in an interval bounded by values of $g(x)$ at consecutive points of the subdivision $\alpha(i)$, $0 \leqslant i \leqslant n$.

The interval $g(\alpha), g(\beta)$ is divided into m parts by the points $g(\alpha(r_j))$, $0 \leqslant j \leqslant m$; if t, T are any two points in the same part, then both t, T lie between the values of $g(x)$ at $x = \alpha(i)$, $x = \alpha(i+1)$, for a certain i, and so, since $g(x)$ is continuous, we can determine x, X between $\alpha(i)$, $\alpha(i+1)$ such that $g(x) = t$, $g(X) = T$.

Hence
$$|f(T) - f(t)| = |f(g(X)) - f(g(x))| < 1/k,$$

which proves that $f(t)$ is continuous in the interval $g(\alpha)$, $g(\beta)$.

We can choose α, β so that $g(\alpha)$, $g(\beta)$ are the lower and upper bounds of $g(x)$ in (a, b), and so the theorem is established. R. L. G.

1968. *A note on identities.*

The identity

$$f(a+ph)-\binom{p}{1}f(a+\overline{p-1}\,h)+\binom{p}{2}f(a+\overline{p-2}\,h)-\ldots$$
$$+(-1)^p f(a)=h^p f^{(p)}(a), \quad \ldots\ldots(1)$$

(where $f(a)$ is a polynomial of degree not greater than p), which is familiar in the calculus of finite differences, admits of an extremely simple verification by the method of partial fractions. For, by Hermite's rule,

$$\frac{f(x)}{(x-a)(x-a-h)(x-a-2h)\ldots(x-a-ph)}$$
$$=\frac{1}{p!\,h^p}\sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r}\frac{f(a+rh)}{x-a-rh}, \quad \ldots\ldots(2)$$

whence, multiplying both sides by $(x-a)(x-a-h)\ldots(x-a-ph)$ and equating coefficients of x^p, the identity is established.

Formula (2) leads also to a variety of novel analogues of the identity (1). For instance, if $f(n)$ is of degree *less* than p, we find the sum of the series

$$\sum_{n\geqslant p+1}\frac{f(n)}{n(n-1)(n-2)\ldots(n-p)}$$
$$=\sum_{n\geqslant p+1}\left\{\frac{f(p)}{n-p}-\binom{p}{1}\frac{f(p-1)}{n-p+1}+\binom{p}{2}\frac{f(p-2)}{n-p+2}-\ldots+(-1)^p\frac{f(0)}{n}\right\}\Big/p!$$
$$=\left\{f(p)\sum_{r\geqslant 1}\frac{1}{r}-\binom{p}{1}f(p-1)\sum_{r\geqslant 2}\frac{1}{r}+\binom{p}{2}f(p-2)\sum_{r\geqslant 3}\frac{1}{r}-\ldots\right\}\Big/p!;$$

and so

$$p!\sum_{n\geqslant p+1}\frac{f(n)}{n(n-1)\ldots(n-p)}=\binom{p}{1}f(p-1)-\binom{p}{2}f(p-2)\left\{1+\frac{1}{2}\right\}$$
$$+\binom{p}{3}f(p-3)\left\{1+\frac{1}{2}+\frac{1}{3}\right\}-\ldots+(-1)^{p+1}f(0)\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}\right\}, \quad \ldots(3)$$

using the identity (1).

Noting that

$$\frac{p-r}{(n-r)(n-r-1)\ldots(n-p)}$$
$$=\frac{1}{(n-r-1)(n-r-2)\ldots(n-p)}-\frac{1}{(n-r)(n-r-1)\ldots(n-p+1)},$$

we find

$$\sum_{n\geqslant p+1}\frac{1}{(n-r)(n-r-1)\ldots(n-p)}=\frac{1}{(p-r)\{(p-r)!\}},$$

whence taking $r=0$, and $f(n)=$ constant, in (3), we have

$$\binom{p}{1}-\binom{p}{2}\left\{1+\frac{1}{2}\right\}+\binom{p}{3}\left\{1+\frac{1}{2}+\frac{1}{3}\right\}-\ldots$$
$$+(-1)^{p+1}\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}\right\}=\frac{1}{p}. \quad \ldots\ldots(4)$$

Taking $f(n) = n(n-1)(n-2)\dots(n-r+1)$, we find

$$\binom{r+1}{1}\binom{p}{r+1} - \binom{r+2}{2}\binom{p}{r+2}\left(1+\frac{1}{2}\right) + \binom{r+3}{3}\binom{p}{r+3}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \dots$$
$$+ (-1)^{p-r+1}\binom{p}{r}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p-r}\right) = \frac{p!}{(p-r)\{(p-r)!\}(r!)}.$$

Incidentally, an identity which was once the subject of several communications to the *Gazette* (" A Problem in Cartophily ", 1939), follows immediately from formula (2). For

$$\binom{p}{1}\frac{f(p-1)}{n-p+1} - \binom{p}{2}\frac{f(p-2)}{n-p+2} + \dots - (-1)^p\frac{f(0)}{n} = \frac{f(p)}{n-p} - \frac{f(n)\,.\,p!}{n(n-1)\dots(n-p)},$$

whence, taking $f(x) = x^p$, $n = p$,

$$\binom{p}{1}(p-1)^p - \binom{p}{2}(p-2)^p\cdot\frac{1}{2} + \binom{p}{3}(p-3)^p\cdot\frac{1}{3} - \dots + (-1)^p\binom{p}{p-1}\cdot 1^p\cdot\frac{1}{p-1}$$
$$= \operatorname*{Lt}_{x\to p}\frac{x(x-1)\dots(x-p+1)f(p) - f(x)\,.\,p!}{x(x-1)\dots(x-p)} = \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p}\right)f(p) - f'(p),$$

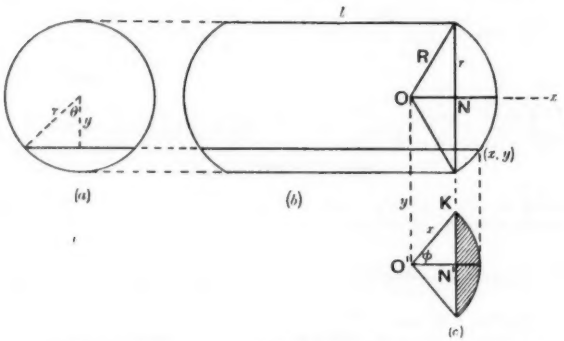
by L'Hospital's rule,

and so $\binom{p}{1}(p-1)^p - \binom{p}{2}(p-2)^p\frac{1}{2} + \binom{p}{3}(p-3)^p\frac{1}{3} - \dots$

$$+ (-1)^p\binom{p}{p-1}1^p\cdot\frac{1}{p-1} = p^p\left(\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p}\right).$$

R. L. G.

1969. *Volume, to a given depth, of a cylindrical tank with spherical caps as ends.*



(*a*) End Elevation. (*b*) Side Elevation : $ON = \sqrt{(R^2 - r^2)}$.

(*c*) Plan at depth y below the centre :

$O'N' = ON = \sqrt{(R^2 - r^2)}$, $N'K = \sqrt{\{x^2 - (R^2 - r^2)\}} = \sqrt{(r^2 - y^2)}$.

For cylindrical part (from end elevation),

$$\text{volume} = lr^2(\theta - \sin\theta\cos\theta)$$
$$= lr^2\cos^{-1}(y/r) - ly\sqrt{(r^2 - y^2)}.$$

Thus if the depth below the centre is h, the volume is

$$V_1 = lr^2 \cos^{-1}(h/r) - lh\sqrt{(r^2 - h^2)}.$$

For a spherical cap,

cross-sectional area at depth y below centre (shaded)

$$= x^2(\phi - \sin\phi\cos\phi)$$

$$= x^2 \tan^{-1}\frac{\sqrt{(r^2 - y^2)}}{\sqrt{(R^2 - r^2)}} - \sqrt{(R^2 - r^2)} \cdot \sqrt{(r^2 - y^2)}$$

$$= (R^2 - y^2)\tan^{-1}\frac{\sqrt{(r^2 - y^2)}}{\sqrt{(R^2 - r^2)}} - \sqrt{(R^2 - r^2)} \cdot \sqrt{(r^2 - y^2)}.$$

Thus the volume of the cap to depth h below the centre is

$$V_2 = \int_h^r \left\{(R^2 - y^2)\tan^{-1}\frac{\sqrt{(r^2 - y^2)}}{\sqrt{(R^2 - r^2)}} - \sqrt{(R^2 - r^2)} \cdot \sqrt{(r^2 - y^2)}\right\} dy.$$

To evaluate this the following definite integrals are needed :

$$I_1 = \int_h^r \frac{dy}{\sqrt{(r^2 - y^2)}} = \cos^{-1}(h/r)\ ;$$

$$I_2 = \int_h^r \sqrt{(r^2 - y^2)} \cdot dy = -\tfrac{1}{2}h\sqrt{(r^2 - h^2)} + \tfrac{1}{2}r^2\cos^{-1}(h/r)\ ;$$

$$I_3 = \int_h^r \frac{y^2\,dy}{\sqrt{(r^2 - y^2)}} = \tfrac{1}{2}h\sqrt{(r^2 - h^2)} + \tfrac{1}{2}r^2\cos^{-1}(h/r)\ ;$$

$$I_4 = \int_h^r \frac{dy}{(R^2 - y^2)\sqrt{(r^2 - y^2)}} = \frac{1}{R\sqrt{(R^2 - r^2)}}\cos^{-1}\frac{h\sqrt{(R^2 - r^2)}}{r\sqrt{(R^2 - h^2)}}.$$

Integrating the first term in V_2 by parts,

$$V_2 = (\tfrac{1}{3}h^3 - R^2h)\tan^{-1}\frac{\sqrt{(r^2 - h^2)}}{\sqrt{(R^2 - r^2)}} + \tfrac{1}{3}\sqrt{(R^2 - r^2)}\int_h^r \frac{(3R^2y^2 - y^4)\,dy}{(R^2 - y^2)\sqrt{(r^2 - y^2)}} - \sqrt{(R^2 - r^2)}I_2. \quad \ldots(\text{i})$$

But

$$\int_h^r \frac{(3R^2y^2 - y^4)\,dy}{(R^2 - y^2)\sqrt{(r^2 - y^2)}} = \int_h^r \frac{\{2R^4 - 2R^2(R^2 - y^2) + y^2(R^2 - y^2)\}\,dy}{(R^2 - y^2)\sqrt{(r^2 - y^2)}}$$

$$= 2R^4I_4 - 2R^2I_1 + I_3.$$

Substituting in (i), doubling, and adding the volume V_1 of the cylindrical part, the whole volume, from the bottom of the tank to a depth h below the centre, is

$$lr^2\cos^{-1}(h/r) - lh\sqrt{(r^2 - h^2)} + \tfrac{4}{3}h\sqrt{\{(R^2 - r^2)(r^2 - h^2)\}} - \tfrac{2}{3}(r^2 + 2R^2)\sqrt{(R^2 - r^2)}\cdot\cos^{-1}(h/r) + 2(\tfrac{1}{3}h^3 - R^2h)\tan^{-1}\frac{\sqrt{(r^2 - h^2)}}{\sqrt{(R^2 - r^2)}} + \tfrac{4}{3}R^3\cos^{-1}\frac{h\sqrt{(R^2 - r^2)}}{r\sqrt{(R^2 - h^2)}}.$$

The last four terms, representing the volume of parts of the spherical caps, may be checked by considering five special cases :

(i) if $h = r$, volume is zero ;

(ii) if $h = 0$, volume is $\pi\{\frac{2}{3}R^3 - (\frac{1}{3}r^2 + \frac{2}{3}R^2)\sqrt{(R^2 - r^2)}\}$, which is equal to the volume of one complete spherical cap ;

(iii) if $h = -r$, the volume is twice as much as in (ii) ;

(iv) if $R=r$, the volume is $\frac{1}{3}\pi(h^3-3r^2h+2r^3)$, which is the volume of a spherical segment of height h;

(v) if $R\to\infty$, the terms in R^3 and R vanish, and the volume is zero.

E. H. LOCKWOOD.

1970. *On the positive and negative sides of a line.*

In elementary analytical geometry, much space is taken up in many of the textbooks in explaining the convention about positive and negative sides of a line. Beginners must find this somewhat complicated convention difficult to learn and to remember, and I have long felt that it is a complication which could quite well be omitted altogether, at any rate from elementary teaching.

However the subject is approached, it is a long matter to describe and explain, and certainly not easy to remember. Probably the quickest way in which to obtain it is to show that, if the join of $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is divided by the line $ax+by+c=0$ in the ratio $m:n$,

$$\frac{m}{n}=-\frac{ax_1+by_1+c}{ax_2+by_2+c};$$

then, if P_1, P_2 are on the same side of the line, $m:n$ is negative, and so

$$ax_1+by_1+c \quad \text{and} \quad ax_2+by_2+c \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

have the same sign. If P_1, P_2 are on opposite sides of the line the expressions (1) have opposite signs. This leads to the statement of the usual convention that the line $ax+by+c=0$ divides the plane of the axes into two compartments such that $ax+by+c$ is positive for all points in one compartment and negative for all points in the other. If c is considered negative (as it is when the equation of the line is put in the form $x\cos\alpha+y\sin\alpha-p=0$), it can be shown that, conventionally, the negative side of the line is the one which contains the origin.

All this is not easy for beginners to grasp, and in any numerical example it is far easier to decide the question by drawing the graph of the straight line and plotting the points!

The same difficulty crops up again in the question of angle-bisectors. To discriminate between

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

it becomes necessary, in theory, to prescribe that the two square roots shall both have positive sign and that the equations be written so that c_1 and c_2 are both negative, so that the origin is on the negative side of both lines. Then, if the + sign of the ambiguity in (2) is taken, the equation represents the bisector of that angle in which the origin lies. This again involves a lot to remember and is bound to lead to confusion in the minds of beginners.

It is not, however, difficult for beginners to see that equation (2) gives one of the bisectors when the + sign is taken and the other when the − sign is taken. Then, whenever a numerical example is set, it is only a matter of a few moments to plot the two given lines, write down the equations of the two bisectors, and identify which bisector is which by observing which has a positive and which a negative gradient.

In these days, when the tendency of the new syllabuses for School Certificate Mathematics is in the direction of simplification, this is surely one of the complications which ought to disappear from all elementary examinations. For example, the "Jeffery" report maintains that finding the factors of expressions such as

$$a^2b+b-ab^2-a \quad \text{and} \quad 8x^2-6x-27,$$

or simplifications of expressions such as

$$\frac{a}{a+b}+\frac{b}{a-b}-\frac{3ab}{b^2-a^2},$$

are too hard for ordinary school-certificate candidates. Although I am not convinced that such examples are so difficult that they should be excluded from ordinary school-certificate papers, I am quite certain that candidates for School Certificate mathematics ought not to be expected to know the convention for positive and negative sides of a line. I would go farther and say that it is relatively so unimportant a convention that I see no justification for its inclusion in any school examination papers, except perhaps in the "Scholarship" type of paper in the Higher School Certificate, and I should not be enthusiastic about its appearing even there. E. G. PHILLIPS.

1971. *The Brocard angle.*

The complex variable is not a very powerful instrument in elementary geometry, for the simple reason that the condition for three points in the Argand plane to be collinear involves in some form or other the introduction of numbers specifically real; complex numbers have to be broken down, or paired with their conjugates, and the result is usually a hybrid algebra that is either vector algebra in a clumsy disguise or a transliteration of work with rectangular coordinates.

My attention having been drawn to the Brocard points by Mr. Piggott, I found an exception to prove the rule. If Ω is a point such that each of the angles ΩCB, ΩAC, ΩBA is ω, then by Ceva's theorem

$$\sin(A-\omega)\sin(B-\omega)\sin(C-\omega)=\sin^3\omega.$$

That is to say, $\cot\omega$ is a root of the equation

$$(\cos A - z\sin A)(\cos B - z\sin B)(\cos C - z\sin C) = -1.$$

But identically

$$\mathrm{Cis}\,A\,\mathrm{Cis}\,B\,\mathrm{Cis}\,C = -1.$$

Hence $-i$ and i are two roots of the cubic equation, and since the sum of these two is zero, the third root is identifiable as the sum of the three roots, and this is the same as the sum of the three roots of the equation

$$(\cos A - z\sin A)(\cos B - z\sin B)(\cos C - z\sin C) = 0,$$

a change in the constant term making no difference to the sum.

The formula for $\cot\omega$ can be written

$$\sin(B-\omega)/\sin\omega = \sin B(\cot A+\cot C) = \sin^2 B/\sin A\sin C = b^2/ac,$$

implying at once that in the triangle CAP in which the angle PAC is ω and the angle ACP is $C+A$, the side CP is ac/b. This is Mr. Steiner's result, that the triangle PBC is indirectly similar to ACB. Since $BP/AC = a^2/b^2$, we can write

$$P = B + (a^2/b^2)(C-A),$$

whence

$$a^{-2}P + (b^{-2}+c^{-2})A = c^{-2}A + a^{-2}B + b^{-2}C,$$

verifying from the cyclic symmetry on the right that the lines drawn in this way through the three vertices are concurrent, and identifying the point of concurrence. E. H. N.

1972. *On Notes* 1727 *and* 1842 (*with* 605): "*Extension of Simson's Line.*"

This is by way of footnote to these Notes, with one specific object, viz. to stress the importance (peculiarly in this context) of the "Complete Angle", or "line-pair", mode of expressing the fundamental Angle-theorems of Geometry.*

1. The facts of Note 605—which is highly condensed—may be re-stated, from the point of view in question, thus (using the figure of Note 1727, p. 107,† but with D—as in Note 605—used for the point there called P'—somewhat conflicting with the notation A, A', etc.):

X, Y, Z are collinear points of BC, CA, AB, respectively; $B'C'$ is perpendicular to BC, at X, etc.: the triangles ABC, $A'B'C'$ being, therefore, (1) *in perspective* (AA', BB', CC' concurrent at P), (2) *directly similar*.

The "similar" system determined by ABC, $A'B'C'$ has the Right Angle as characteristic Complete Angle (of each line-pair of the correspondence); and its "double-point", D, is the second intersection of the circles ABP, $A'B'P$—from the Complete Angle relations

$$(AB, AD) \equiv BAD \equiv BPD \equiv B'PD \equiv B'A'D \equiv (A'B', A'D),\ddagger$$

whence (interpolating $(AB, AC) \equiv (A'B', A'C')$),

$$(AC, AD) \equiv (A'C', A'D), \text{ etc.},$$

(this establishing the "double-point" property—which can clearly not hold for more than one point D).

And the two points P and D—which are associated, respectively, with (1) and (2) above—are thus the two points of intersection of the circumcircles of the triangles ABC, $A'B'C'$ (circles which, from the similarity, must intersect orthogonally).§

2. The facts of § 1 of Note 1842 take the following Complete Angle form:

if PS is perpendicular to BC and S is also on the circle ABC,

$$(PS, AS) \equiv PSA \equiv PCA \equiv C'CY \equiv C'XY \equiv (B'C', XY).\ddagger$$

Therefore, since PS is parallel to $B'C'$, the transversal XYZ is parallel to Simson's line for P, with respect to the triangle ABC; and so, again, to Simson's line for P, with respect to the triangle $A'B'C'$. Whereas the two Simson's lines for D—being corresponding lines of the similarity—are perpendicular to one another.

The keynote of this contribution is *generality* and *simplicity* of argument.

D. K. PICKEN.

* See XI, 161, pp. 188–193, especially § 11 (with *Proc. London Math. Soc.*, (2), XXIII, Pt. 1). The facts dealt with in the Notes are integral to that discussion—which missed them, while including (§ 11 (v)) consideration of the other type of Simson's line theory converse (concurrent perpendiculars). *See also* the 1923 (first) *Report on the Teaching of Geometry*, pp. 11 and 66.

† Even one such diagram is a *tour de force*, which comes to its own when associated with "generality" of argument. That is what this Note is meant to supply. See also *loc. cit.*, § 2, Note.

‡ *loc. cit.*, §§ 3–8. For the circle-theorem, *see also* my Note 1899.

§ Alternatively, the well-known theorem that the angle of inclination $\equiv ADA'$ has line-pair proof as follows: if t, t' denote the tangents at D, and c the common secant DP,

$$(t, c) \equiv (AD, AP) \equiv (AD, AA')$$

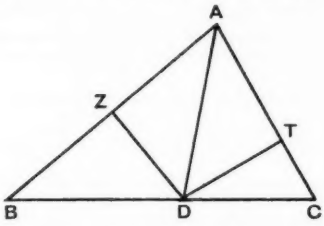
and $$(c, t') \equiv (AA', A'D).$$

Hence $$(t, t') \equiv ADA'.$$

1973. *On Euclid* VI. 3.

Textbook proofs of this proposition (that the bisectors of the vertical angle of a triangle divide the base in the ratio of the sides, and conversely) seem to use one of two constructions : (i) a parallel is drawn to the bisector, through one base point ; (ii) perpendiculars are dropped on the bisector from both base points. The former is far more stylish, but not easily remembered by average students. A third method, which I have not seen anywhere, but which seems both elegant and easily assimilated in the usual course on similar figures relies on simple area properties, and brings in the fundamental property of a bisector of an angle directly.

If in the triangle ABC, AD is the bisector, and perpendiculars DT, DZ are



drawn from D to the opposite sides, then it follows that since $DZ = DT$,

$$\text{area } ABD : \text{area } ACD = AB : AC,$$

and also that since the triangles have a common altitude :

$$\text{area } ABD : \text{area } ACD = BD : DC.$$

The converse and the application to the external bisector are equally simple. I am puzzled as to why this method is not more generally used.

J. I. Mason.

1974. *Right circular cone.*

I. The student of *Solid Geometry*, after studying the properties of the cone represented by the homogeneous equation of the second degree has to wait till he comes to the article on " Conicoids of Revolution " to find the condition that the above equation may represent a right circular cone. Of course, he may change over to the new axes by the scheme given in Bell's *Co-ordinate Geometry of Three Dimensions*, Art. 52, taking one of the new axes to be the axis of the cone.

In what follows we obtain the result by a direct comparison.

Since the generators make the same angle (α) with the axis $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ of the cone (l, m, n being actual direction cosines), equation of the right circular cone is

$$(lx + my + nz)^2 = (x^2 + y^2 + z^2)\cos^2\alpha,$$

i.e. $x^2(l^2 - \cos^2\alpha) + y^2(m^2 - \cos^2\alpha) + z^2(n^2 - \cos^2\alpha) + 2lmxy + 2lnxz + 2mnyz = 0.$

Comparing this with the general equation of the second degree we have

$$\frac{a}{l^2 - \cos^2\alpha} = \frac{b}{m^2 - \cos^2\alpha} = \frac{c}{n^2 - \cos^2\alpha} = \frac{h}{lm} = \frac{g}{ln} = \frac{f}{mn},$$

which gives the results :

$$(1) \quad l : m : n = \frac{1}{f} : \frac{1}{g} : \frac{1}{h}$$

and $$(2)\quad \frac{b-c}{f^2}+\frac{c-a}{g^2}+\frac{a-b}{h^2}=0,$$

provided none of f, g, h is zero.

(2) can be written in the mnemonic form :

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f^{-2} & g^{-2} & h^{-2} \end{vmatrix}=0.$$

We can express the semi-vertical angle also in terms of the coefficients a, b, c, etc.

II. In Conic Sections we have the result that the intersection of a right circular cone by a plane making an angle $\gtrless$ semi-vertical angle, with the axis of the cone, decides the nature of the conic section as ellipse, parabola or hyperbola.

Because of the elegance of the analytical method, I add an analytical proof for the result.

There is no loss of generality in taking the equations of the cone and the plane to be

$$ax^2+by^2+cz^2+2hxy+2gxz+2fyz=0 \text{ and } z=p.$$

If the plane makes with the axis $x/l=y/m=z/n$ an angle $\phi \gtrless \alpha$ (the semi-vertical angle),

$$\sin\phi \gtrless \sin\alpha,$$

i.e. $$n^2 \gtrless \sin^2\alpha, \quad \text{or } \cos^2\alpha-(1-n^2)\gtrless 0,$$

i.e. $$\cos^4\alpha-(l^2+m^2)\cos^2\alpha \gtrless 0, \quad \text{since } l^2+m^2+n^2=1,$$

i.e. $$(l^2-\cos^2\alpha)(m^2-\cos^2\alpha)\gtrless l^2m^2,$$

i.e. $$ab \gtrless h^2;$$

i.e. the conic $ax^2+by^2+2hxy+2gxp+2fyp+cp^2=0$ is an ellipse, parabola or hyperbola.

S. PARAMESWARAN.

1975. *Shortcutting in Multiplication on a Calculating Machine.*

Every operator of one of the usual hand-calculating machines knows that to multiply by 9 he has to make one turn in the tens position and one negative turn in the units position, thus making only two turns altogether. This can be denoted, with obvious meaning, by $9=1\bar{1}$, where the bar above a number indicates negative turns. Similarly, we write, say, $86=1\bar{1}4$ and $95=10\bar{5}$.

Any experienced computer finds these shortcuts for himself and it is, of course, of no consequence if he makes a few more turns than absolutely necessary. However, it may be of theoretical interest to investigate which procedure leads in every case to the minimum of turns and the following set of rules is suggested, with an invitation to communicate possible improvements.

(*a*) The figures 0, 1, 2 and 3 are called low figures.
(*b*) The figures 6, 7, 8 and 9 are called high figures.
(*c*) 5 followed or preceded by a high figure is high, otherwise it is low.
(*d*) 4 is high if preceded *and* followed by a high figure, otherwise it is low.
(*e*) 55 between low figures,
45 preceded by a high figure and
44 between high figures
can be taken as high or as low. We call these combinations " neutral ".

But as far as following or preceding is relevant under definitions (*a*) to (*d*), neutral combinations operate like high figures. If they result in making a neutral or an otherwise low figure high, then they must themselves be treated as high.

NOTE.—It follows from (*a*) that 555 is high, because the neutral 55 makes the following 5 high. Moreover, 45 between high figures is necessarily high, because the 5 is high under (*c*) and then the 4 under (*d*).

RULES :

Low figures are reached by successive additions.

A *run* of *high* figures is approached by

(i) adding one to the preceding figure (which is built up by addition). This may make it necessary to use one more position than the number of digits in the run, *e.g.* $8 = 1\bar{2}$.

(ii) subtracting, in respect of all following figures except the last one, their complement to 9, and

(iii) reaching the last one by subtracting its complement to ten.

A few examples will show the working of these rules. We have

(1) $1846 = 2\bar{1}5\bar{4}$, under (*d*). This rule is frequently disregarded, even in a popular booklet on machine calculation.

(2) 65455. The 5 after 6 is high (*c*). The last 55 is neutral, but in its effect on the 4 it is high (*e*). Thus the 4 is itself high (*d*). Hence we have $65455 = 1\bar{3}4\bar{5}45$.

(3) 6447. Here we have 44 between high figures, so that it is neutral. We can therefore take it as high, which leads to $1\bar{3}5\bar{5}3$, or as low, giving $1\bar{4}4\bar{5}3$. The number of turns is in both cases 17.

(4) 84455. This is a more complicated example. The 55 is neutral. Let us then, first, take it as low. The 44 is also low and 8 is of course high. We have thus $1\bar{2}4455$ with 21 turns altogether. On the other hand, 55 might also be high. Then 44 becomes neutral. Let it, again, be low. This produces $1\bar{2}45\bar{4}5$, again 21 turns. Or, let 44 be high. Then we obtain $1\bar{1}5\bar{5}45$ with the same total of turns.

(5) Similarly, let us investigate 94545. Here the first 45, following a high figure, is neutral. But as far as the effect on the second 45 is concerned, this becomes also neutral. Hence there result the following three possibilities, with 20 turns in every case: $10\bar{5}4\bar{5}5$, $10\bar{5}5\bar{4}5$, $1\bar{1}4\bar{5}45$.

Finally, a further problem may be mentioned. The number of minimum turns for the production of any number can be considered as a function of this number. A study of this function might lead to interesting results. It would, in particular, be useful to have a simple rule to decide which of two factors needs a smaller number of turns. Obviously this would be the factor to be brought into the multiplier register and the other factor would be set on the keyboard. S. VAJDA.

1976. "*Why does a bicycle keep upright?*" (*Math. Gazette*, Vol. XXX, No. 291). Since the publication of the above article, my attention has been called to a paper by F. J. W. Whipple (*Quarterly Journal*, 30 (1899)). This paper, like that of Carvallo, attacks the problem analytically, arriving also at the result that a bicycle ridden without touching the handlebars is stable for a band of velocities. F. G. M.

REVIEWS.

Methods of Mathematical Physics. By HAROLD JEFFREYS and BERTHA SWIRLES JEFFREYS. Pp. viii, 679. 63s. 1946. (Cambridge University Press)

In a recent number of the *Mathematical Gazette*, the editor claimed that his reviewers actually read the books they review and speak with acknowledged authority on them. The present reviewer only scores 50 per cent. in this test; he has read the book, but is certain that no one, except its authors, could speak with authority on the whole of it. It seems therefore preferable that he should try to describe the aim and contents of the book as a whole, at the risk of making the review a rather dull catalogue.

The immediate stimulus for writing the book was the fact that Professor Jeffreys' well-known Cambridge tract on *Operational Methods in Mathematical Physics* went out of print, and it was decided to write a larger work giving an account of those parts of pure mathematics that are most frequently needed in physics. Naturally a book containing all the methods used in different branches of physics would have been impossibly long, and some selection had to be made. No attempt has been made to give a detailed account of any branch of physics: that is a matter for the special textbooks.

The first thing which will strike the reader is the standard of rigour and generality aimed at. The authors do not accept the common view that any argument is good enough if it is intended to be used by scientists. They feel that it is as necessary in theoretical physics as in pure mathematics that the fundamental principles should be clearly stated and that the conclusions should follow logically from them. For this reason, mathematical theorems are here always proved rigorously, though sometimes under conditions which are sufficient for the purpose but which would be regarded as unnecessarily restrictive by the pure mathematician.

The next is that there is hardly any overlapping with Courant and Hilbert's *Methoden der mathematischen Physik* and very little with Bateman's *Partial Differential Equations of Mathematical Physics*. The reason is that the Jeffreys' book deals with more elementary and more fundamental mathematics, whereas the books by Bateman and Courant-Hilbert are more specialised and are concerned with the actual solution of the boundary value problems which arise in connection with the differential and integral equations of physics.

The book starts with an excellent introduction in Chapter 1 (pp. 1–48) to the theory of functions of a *Real Variable*. This is followed by Chapters 2 (pp. 49–77) and 3 (pp. 78–105) which deal with *Scalars and Vectors* and with *Tensors*. The theory of tensors is developed much on the lines of Professor Jeffreys' book on *Cartesian Tensors*; and the resulting practice in using the new notation in a simple and restricted form is a valuable introduction to the more general theory. Chapter 4 (pp. 106–154) deals with *Matrices* in some detail, including the application of the theory to the simultaneous reduction of two quadratic forms to sums of squares and the corresponding theory of small oscillations in dynamics. The chapter concludes with an account of the effect on vector and tensor analysis of using oblique cartesian or curvilinear coordinates, and of the electromagnetic tensor of special relativity.

In Chapter 5 (pp. 155–174), the theory of *Multiple Integrals* is developed, with the associated theorems of Green and Stokes, and is followed in Chapter 6 (pp. 175–203) by an excellent account of the fundamentals of *Potential Theory*.

Most of the material in Professor Jeffreys' tract on *Operational Methods* appears in Chapters 7, 8 and 12, though the methods are also used in subsequent chapters, notably in connection with Bessel functions. Chapter 7 (pp. 204–219) gives the elementary theory, Chapter 8 (pp. 220–236) some *Physical applications*

of the Operational Method, and the Bromwich form of the theory is deferred until Chapter 12.

Chapter 9 (pp. 237–285) is somewhat of an innovation. The modern mathematical physicist does not expect Nature to consist entirely or even largely of problems designed to come out neatly in finite terms, and has to be prepared to accept approximate solutions obtained by *Numerical Methods*. Here we find a good account of the methods of interpolation, of numerical integration, and of numerical solution of differential equations.

Chapter 10 (pp. 286–304) deals with the *Calculus of Variations* in its classical form, in particular with its application to analytical dynamics.

A surprisingly complete account of the theory of *Functions of a Complex Variable* will be found in Chapter 11 (pp. 305–346), followed in Chapter 12 (pp. 347–379) with the application to *Contour Integration and Bromwich's Integral*. Chapter 13 (pp. 380–399) on *Conformal Representation* is mainly concerned with transformations where the boundaries are straight lines, circles or conics, but also discusses the Joukowsky and Schwarz-Christoffel transformations.

Chapter 14 (pp. 400–432) is concerned with *Fourier's Theorem*. The convergence of the Fourier series of a function of bounded variation, the summability by Abel's method of the Fourier series of an integrable function and the questions of the differentiation and integration of a Fourier series are all discussed very clearly. This is followed by an account of the theory of Fourier's integral of a function of bounded variation.

In Chapter 15, entitled *The Factorial and Related Functions* (pp. 433–444), we meet the Gamma function in a somewhat disguised form. The authors prefer to write $z!$ for $\Gamma(z+1)$ for any value of z on the ground that the extra 1 in the definition of the Γ function is a minor but continual nuisance. Whilst there is much to be said for economy in notation, the extraneous 1 cannot be suppressed quite so easily : it crops up again in the Beta integral.

$$\int_0^1 t^m(1-t)^n\,dt = \frac{m!\,n!}{(m+n+1!)}\,.$$

And there is, in any case, much to be said for a notation which distinguishes between formulae valid for all values of z and those true only for positive integral values of z.

The Solution of Linear Differential Equations of the Second Order by the power series method is discussed in Chapter 16 (pp. 445–467). *Asymptotic Expansions* are introduced in Chapter 17 (pp. 468–496) where the method of steepest descents is clearly explained and is used to obtain Stirling's approximation and the asymptotic formulae for the Airy functions and to explain the theory of dispersion. The chapter concludes with an account of the asymptotic solution of linear differential equations, including the Jeffreys' method commonly known in quantum mechanics as the W.K.B. method.

The method of separation of variables is applied to *The Equations of Potential, Waves and Heat Conduction* in Chapter 18 (pp. 497–513), and the spherical harmonics are first introduced here. *Waves in One Dimension and Waves with Spherical Symmetry* are discussed in Chapter 19 (pp. 514–530) when the remarks on the differentiability conditions in the solution of the problem of the vibrating string are of especial interest. The operational solution of the one dimensional wave equation with fixed ends is shown to lead to the Fourier solution and to the solution in finite terms by using different methods of interpretation. Problems concerned with the *Conduction of Heat in One and Three Dimensions* are solved in Chapter 20 (pp. 531–541) again by the operational method.

The rest of the book is concerned with various types of special functions, *Bessel Functions* (Chapters 21 and 22, pp. 542–573), the *Confluent Hyper-*

geometric Function (Chapter 23, pp. 574–595), *Legendre Functions and Associated Functions* (Chapter 24, pp. 596–631), and *Elliptic Functions* (Chapter 25, pp. 632–655).

Needless to say, the book is written in an interesting and clear style and every chapter but two is headed by an apt quotation in the way Professor Jeffreys has led us to expect: one wonders why he failed with Chapters 19 and 22. The book is intended to be used not only as a book of reference (it will certainly prove a valuable supplement to Whittaker and Watson, Bateman and Courant-Hilbert), but also as a textbook, containing well-chosen examples taken from the Mathematical Tripos and the Final Honours Examinations at Manchester and Imperial College.

We are grateful to the authors for the excellent way they have carried out a seemingly impossible task and wish their book every success. E. T. COPSON.

Calculating machines. Recent work and prospective developments. By D. R. HARTREE. Pp. 40. 2s. 1947. (Cambridge University Press)

This booklet consists of Professor Hartree's inaugural lecture on his appointment as Plummer Professor in Mathematical Physics at the Cavendish Laboratory in succession to the late Sir Ralph Fowler. It contains a brief description of the effect of electronic methods on automatic computing. Professor Hartree has recently had the opportunity of working with the ENIAC (Electronic Numerical Integrator and Calculator) at the University of Pennsylvania, and without discussing technical details he gives us a very clear account of the tasks it is able to undertake.

The most obvious advantage of electronic calculating machines is of course their speed, while their most severe limitation at present is their memory capacity. When the latter has been increased sufficiently, the solution of many problems will become possible which at present are impracticable simply on account of the arithmetical labour involved. The lecture concludes with some speculations as to the future developments in this field. B. M. B.

The Theory of Matrices, Determinants and Invariants. By H. W. TURNBULL. 2nd edition. Pp. xvi, 368. 30s. 1946. (Blackie)

The last century saw great advances in Algebra in general, and in Invariant-theory in particular, both on the continent, and in England under the leadership of Cayley and Sylvester. Early this century a strong reaction set in and the subjects fell very markedly from popularity. This change was partly due to the vagaries of fashion, but partly perhaps, because in the wealth of attention, which had been lavished previously on the subjects, the working out of details had outstripped the development of fundamental theory, so that these subjects appeared worked out, with fertility exhausted.

Whatever justification may have existed for a transfer of attention to other branches of mathematics, it certainly did not justify the almost complete neglect of Algebra and Invariant-theory which followed and which has lasted almost until present times, for these subjects now present a field as rich and fertile for research as any other.

Under such circumstances a peculiar responsibility rests on the faithful few who follow the subject, once so dominant, lest the hard-won learning of one generation should be buried by antiquity and lost to the next. The loss in this case would be the more regrettable owing to the wealth of applications of Invariant-theory in Geometry and in Applied Mathematics.

Hence the publication in 1929 of this book by Professor Turnbull, the most notable exponent of the subject in this generation, was an event of some importance, and this reissue of the second edition should not pass without notice and tribute.

Professor Turnbull introduces the subject in an original way, through the theory of determinants and matrices, and the technique of these plays a correspondingly predominant part in the development. But this work on matrices and determinants goes far beyond that of a mere introduction to Invariant-theory. Eight chapters are devoted to this section, and these go very deeply into the theory. Many elaborate techniques are displayed, among which may be remarked especially a permutation summation indicated by dots, which is effectively used throughout the book.

All aspects of classical Invariant-theory, including symbolic and non-symbolic methods, are described from the author's individual approach. There are many helpful descriptions and references to more recent developments.

The emphasis on matrices and determinants is maintained in the new material which has been added to the second edition. The first of two new chapters contains many interesting properties of determinants, of which perhaps the most notable concerns *bideterminants*. It is good to see some account, albeit brief, in the last chapter, of Young's standard forms and tableaux, a remarkable contribution to the theories of groups and of invariants by the late Alfred Young.

D. E. LITTLEWOOD.

Introduction à la Topologie Combinatoire, vol. I, Initiation. By MAURICE FRÉCHET and KY FAN. Pp. viii, 88. 1946. 8s. 6d. (Librairie Vuibert)

Since nearly all the existing books on combinatory topology are given in a more or less abstract form, modern topologists have long wished for a pretty introduction to the subject in an intuitive and picturesque style, especially suitable for the youngsters and non-specialists. One of the authors, Prof. Fréchet, perceived this gap and tried to fill it by giving a series of public lectures in 1935 at the Sorbonne, Paris. This idea of Fréchet has now been realised by the collaboration of Dr. Ky Fan.

The first volume of this book has already appeared. It contains three easily intelligible chapters which can be read with ease by an undergraduate having a sufficient knowledge of elementary geometry. Its main object is to lead the reader, supposed to be a beginner, to the historical interest of topology. It seems to the reviewer that this book is a very good preliminary one for a student who is going to read Prof. M. H. A. Newman's treatise, *Elements of the Topology of Plane Sets of Points*.

The first chapter describes what topology is in the most elementary language. It starts from the interesting classical problems not yet solved; *i.e.* the four-colour problem and the problem of neighbouring regions. In the sections 4–6, a vivid definition of this new branch of mathematics is formulated; and then a comparison between elementary geometry, projective geometry, and topology is given in §7 to clarify the foregoing definition. In §8, the absolute and relative topological properties are distinguished with the linking coefficient as an illustration; and the notions of isotopy and homotopy are introduced. In §9, a brief description of the two main branches of topology, *e.g.* the set-theoretic and the combinatory topology, is given.

The remaining two chapters consist of an elementary presentation of the theory of surfaces. It commences from the historical Descartes-Euler theorem (§11), and then defines the characteristic of a surface (§13). In §§14 and 15, the properties of being one-sided or two-sided and of being orientable or non-orientable are discussed in a figurative way. In the sections 16–18, the construction of surfaces from a polygon by the operation of identification is studied; and the topological definition of closed surfaces is given in §19. The remaining sections contain a detailed exposition of the classification theory of surfaces.

S. T. HU.

THE ANNALS OF THE COMPUTATION LABORATORY OF HARVARD UNIVERSITY.

1. Vol. I. **A Manual of Operation for the Automatic Sequence Controlled Calculator.** Pp. xvi + 561 + 17 plates. 1946.

2. Vol. II. **Tables of the Modified Hankel Functions of Order One-third and of their Derivatives.** Pp. xxxvi + 235. 1945.

3. Vol. III. **Tables of the Bessel Functions of the First Kind of Orders Zero and One.** Pp. xxxviii + [652]. 1947.

4. Vol. IV. **Tables of the Bessel Functions of the First Kind of Orders Two and Three.** Pp. x + [652]. 1947.

By the STAFF OF THE COMPUTATION LABORATORY; Cambridge, Mass.: Harvard University Press; London: Geoffrey Cumberlege, Oxford University Press. $10''{\cdot}4 \times 7''{\cdot}8$. 55s. each volume.

The series of volumes of which these are the first four has its origin in the construction of the I.B.M. Automatic Sequence Controlled Calculator by the International Business Machines Corporation under the inspiration of Commander Howard H. Aiken, U.S.N.R., of Harvard University, and incorporating many inventions of the foremost I.B.M. engineers. On completion, the machine was presented to Harvard University and immediately leased to the U.S. Navy at $1 a year for war work. In the intervals between urgent war jobs certain mathematical tabulations were commenced and are now beginning to appear in the final printed form.

1. The first volume describes the machine itself, and gives full details of its operation on the various standard arithmetical and mathematical processes that arise often in the evaluation of numerical results.

The machine contains 72 storage counters, each with a capacity of 23 digits and a sign. For smaller numbers counters may be split into two, while for larger numbers, certain pairs can be combined into single 46 digit units. There are also 60 switch-set 24-figure registers for holding constants; these also can be split. Multiplication and division are carried out by means of two special units.

Three special units are for calculating logarithms, antilogarithms (or exponentials) and sines. These are computed by means of the appropriate series, after some initial preparation involving the use of short tables—60 to 70 values in all—stored in the machine. Thus all logarithmic, exponential, circular and hyperbolic functions are accessible without use of printed tables. Other functions needed may be supplied to the machine via tapes or cards.

The most important feature of the machine is the control tape, which is code-punched in three sections. The first instructs the machine where to find its data; the second tells it where to put the result, on completion of the operation dictated by the third section of the tape. The third section also contains the instruction to the machine to continue operation, using the next line of code on the tape; a run of the machine can be broken by omission of this instruction to continue, or by a failure of a check applied to the calculations in the machine. Control tapes may often be stored and used again and again, being independent of the actual figures used in the operation. Continuous rings of control tape may be used on repetitive calculations.

An interesting special point is that it is possible to read from any storage counter the absolute value of the quantity therein; it is also possible to prefix to this the sign of the quantity in another storage counter (by means of a particular counter that is equipped with special relay circuits). Thus evaluation of an odd-function, say $f(x)$, may be confined to positive x, since

$f(-x) = -f(x)$ may be readily deduced. Likewise, a discontinuous function may be dealt with, using, for example, $(x+|x|)/2x$ to represent 0 when $x<0$ and 1 when $x>0$. Another use is in applying checking processes; these may always be reduced to the form that the calculated quantity c shall be numerically less than a selected positive tolerance t. The check thus consists in testing the sign of $t-|c|$, and the "continue operation" instruction can be made to depend on the result of this test.

The time taken to perform various operations on the machine is of interest. Additions and subtractions each take about $\frac{1}{8}$ second, 23-figure multiplications about 3 seconds and division twice as long. A full capacity logarithm, antilogarithm or sine takes about a minute. These times do not compare with those of electronic machines (in which multiplication may take only a fraction of $\frac{1}{1000}$ second), but the I.B.M. machine has a great advantage over ordinary computors because of its 24-hour day (without breaks for tea!). It is the first fully automatic calculator in the field and (even though speed of calculation is substantially greater in electronic machines) produces its results much faster than they can conveniently be published.

Tables produced by the machine may be typed by an electromatic typewriter. This gives typescript perfectly uniform in quality, admirably suited for reproduction by photolithography, with all the consequent reduction in chances of error, and in the saving of work of transcription, composition and proof-reading.

The book gives full details of the working of the machine, and of its mode of use for various processes, and describes various control tapes.

A 65-page bibliography of numerical analysis is a useful feature of the volume, which may otherwise be of interest only to a rather limited class of reader or user.

2. These "Tables of the Modified Hankel Functions of Order One-third and of their Derivatives" form the first volume of tables to be published as a result of the work of the machine described above. They were computed in the time the machine had to spare from urgent war work, and give a worthy start to the series of Annals (this volume was the first to be published, and was even held back for a time in order not to be too far in advance of Vol. I).

The functions tabulated, $h_1(z)$ and $h_2(z)$, are defined by

$$h_1(z) = \frac{k}{i\pi}\int_{L_1} \exp(\tfrac{1}{3}t^3 + zt)\,dt, \qquad h_2(z) = \frac{k^*}{-i\pi}\int_{L_2} \exp(\tfrac{1}{3}t^3 + zt)\,dt \ldots\ldots (1)$$

where

$$k = 12^{1/6}e^{-i\pi/6},\ k^* = 12^{1/6}e^{i\pi/6},$$

and the contours L_1 and L_2 run from $z = -\infty$ to $z = e^{\pm i\theta}\times\infty$ respectively, with $\frac{1}{6}\pi < \theta < \frac{1}{2}\pi$. These functions satisfy the differential equation

$$d^2u/dz^2 + zu = 0 \quad \ldots\ldots (2)$$

It can be verified that

$$h_1(z) = (\tfrac{2}{3}z^{3/2})^{1/3}\,H^{(1)}_{1/3}(\tfrac{2}{3}z^{3/2}), \quad h_2(z) = (\tfrac{2}{3}z^{3/2})^{1/3}H^{(2)}_{1/3}(\tfrac{2}{3}z^{3/2})$$

in which $H^{(1)}_{1/3}(T)$ and $H^{(2)}_{1/3}(T)$ are the usual Hankel functions $J_{1/3}(T) \pm iY_{1/3}(T)$. This identification is the origin of the title to the volume.

In fact, it seems to the writer that the inconvenient argument $T = \frac{2}{3}z^{3/2}$ of the Hankel functions renders this identification less useful than the following. In B.A. Mathematical Tables, Part-Volume B†, page B9, two functions are defined which satisfy the differential equation $y'' = xy$. These are

† J. C. P. Miller. *The Airy Integral, giving Tables of Solutions of the Differential Equation* $y'' = xy$. Cambridge University Press, 1946.

$$\mathrm{Ai}(x)=\frac{1}{2\pi i}\int_{L_2-L_1}\exp(\tfrac{1}{3}t^3-xt)\,dt=\frac{1}{\pi}\int_0^\infty\cos(\tfrac{1}{3}t^3+xt)\,dt,$$

$$\mathrm{Bi}(x)=\frac{1}{2\pi}\int_{L_2+L_1}\exp(\tfrac{1}{3}t^3-xt)\,dt=\frac{1}{\pi}\int_0^\infty\{\exp(-\tfrac{1}{3}t^3+xt)+\sin(\tfrac{1}{3}t^3+xt)\}\,dt.$$

Thus

$$h_1(z)=-i\,.\,2\,.\,12^{1/6}\,\mathrm{Ai}(-\omega z)=k[\mathrm{Ai}(-z)-i\mathrm{Bi}(-z)],$$
$$h_2(z)=-i\,.\,2\,.\,12^{1/6}\,\mathrm{Ai}(-\omega^2 z)=k^*[\mathrm{Ai}(-z)+i\mathrm{Bi}(-z)].$$

in which $\omega=e^{2\pi i/3}$ is a cube root of unity. (The second form in all four formulae is given in the introduction to the volume under review.)

The functions tabulated, $h_1(z)$ and $h_2(z)$, are thus constant multiples of Airy Integrals, but since the argument z has a rectangular grid of values, based on the axes $x=0$, $y=0$, the corresponding Airy Integrals are given for a rectangular argument based on the lines $y=\pm\sqrt{3}x$, $x=\mp\sqrt{3}y$. These tables are thus to some extent complementary to those of P. M. Woodward and A. M. Woodward, etc. (*Phil. Mag.* (7) **37**, 236–261, 1946), which give four-decimal tables of $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$ and their first derivatives for a rectangular grid $x, y=-2{\cdot}4(0{\cdot}1)+2{\cdot}4$.

The Harvard tables give real and imaginary parts of $h_1(z)$, $h_1'(z)$, $h_2(z)$, $h_2'(z)$ to eight decimals for $y=0(0{\cdot}1)Y$, $x=-X(0{\cdot}1)+X$; $|X+iY|<6$†. Values for negative y may be obtained by means of complex conjugates. The use of a rectangular grid over an essentially circular domain is a little unusual. It is perhaps worth noting that relations between functions with arguments z, ωz and $\omega^2 z$ make it a simple matter to obtain from the tables the functions along rays from the origin for every 15° of arg z. Such relations include

$$h_1(z)=h_2(\omega^2 z),\qquad h_2(z)=h_1(\omega z),$$
$$h_1(z)+\omega h_1(\omega z)+\omega^2 h_1(\omega^2 z)=h_2(z)+\omega h_2(\omega z)+\omega^2 h_2(\omega^2 z)=0.$$

Part I of the Introduction gives an interesting and full account of "The Modified Hankel Functions and their Properties" and includes diagrams giving contours in the x-y plane for $\Re(h_1)=$ constant and for $\Im(h_1)=$ constant (the symmetry about the line $y=\sqrt{3}x$ is striking and makes the asymmetry of the argument-grid very obvious), and graphs of $\Re$ and $\Im$ parts of h_1, along the lines $x=0$, $y=0$ and $y=\sqrt{3}x$. Part II gives "The Method of Computation of the Function." This is followed by a Bibliography of 41 items (rather surprisingly, the name of H. Jeffreys, to whom the definitions of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are due, does not occur in this list). Four short tables follow giving coefficients in ascending and asymptotic series for $h_1(z)$ and $h_2(z)$, and useful constants.

Although the processes of computation and methods of reproduction used for this volume make it almost a waste of time to apply numerical checks, it has, nevertheless, seemed worth while to difference several hundred values on a National accounting machine, as this table is the first of a new series. No error was found in this way.

3 and 4. These massive volumes are very similar. They give values of the Bessel Functions $J_n(x)$ for $n=0$ and 1 (Vol. III), 2 and 3 (Vol. IV) to 18 decimal places for $x=0(0{\cdot}001)25(0{\cdot}01)99{\cdot}99$.‡ No differences are given. The former (alone) has a good Introduction defining the functions tabulated, the computation of the tables, and interpolation in the tables. Auxiliary tables give 50-figure values of coefficients in the series for J_0, J_1 and J_2 in ascending

† Roughly; more precisely $x^2+y^2\leqslant 36{\cdot}56$, with 18 further pairs of values $\pm x+iy$ (of a possible 26) with $36{\cdot}56<x^2+y^2\leqslant 37{\cdot}06$.

‡ The omission of argument 100 seems most unfortunate and unnecessary.

powers of x, and 40-figure values of coefficients for J_3; 30-figure coefficients in the asymptotic expansions for J_0, J_1 and J_2 are also given.

In the Introduction it is remarked that "it was felt that the enormous volume of computation being undertaken by this laboratory warranted the construction of completely independent tables". The tables were therefore computed completely from scratch, and without reference to earlier tables, apart from intercomparisons during the final stages of the work.

General remarks. All these volumes have been produced by photographic methods from typescript prepared, in the case of the main tables, by the Tabulator itself by means of an electromatic typewriter. The results are very good and perfectly uniform and even in density; even so they are less satisfactory than printing from type. Volumes 3 and 4 have a very good appearance, and are very easy to read, in spite of the use of figures of uniform height. In fact, the figures used in Vol. 2 are better, having heads and tails, and are easier to read, even though the reproduction is a little less good, and the figures a little less black. The figures used in Volume 2, combined with the reproduction of Volumes 3 and 4 would have given an excellent result.

The price of these volumes is high, particularly in the case of the rather small Volume 2. It is perhaps not excessive for volumes the size of Volumes 3 and 4, although it should be noted that reproduction of typescript in a conveniently legible size results in rather a small amount of material to a page, and so to massive volumes which may contain a relatively small amount of material. Thus, by printing, the contents of Volumes 3 and 4 might have been combined into a single volume of the same size, which would yet have been as easy to read, since figures printed in type may be smaller than those reproduced from typescript and yet equally easy or easier to read. When the full tabulation of Bessel Functions that is contemplated—with order n and argument x both up to 100—is complete, the result will be over a dozen heavy volumes, and it might often be easier to interpolate, even with large second differences, into a compact table than to pick out the appropriate volume or volumes and find the appropriate pages.

Nevertheless, the volumes now reviewed are of considerable interest and usefulness and, except possibly for the first, should be in all libraries making a speciality of mathematical tables. J. C. P. MILLER.

THE MATHEMATICAL TABLES PROJECT. V.

1. **Tables of Fractional Powers.** Pp. xxx, 490. $7\frac{1}{2}'' \times 10\frac{1}{2}''$. 1946. 45s. (Columbia University Press, New York)

2. **Tables of Spherical Bessel Functions.** Volume 1. Pp. xxviii, 380. $7\frac{1}{2}'' \times 10\frac{1}{2}''$. 1947. 45s. (Columbia University Press, New York)

3. **Table of the First Ten Powers of the Integers from 1 to 1,000.** Pp. 80. $8\frac{1}{2}'' \times 14''$. 1939. Out of print. (Federal Works Agency. W.P.A. for the City of New York)

[All the tables were prepared by the Mathematical Tables Project, (formerly known as the New York Mathematical Tables Project) conducted under the sponsorship of the National Bureau of Standards; technical director, Arnold N. Lowan. British Agents, Scientific Computing Service, 23 Bedford Square, London, W.C.1.]

In four earlier articles (*Mathematical Gazette*, **29**, 1945, pp. 29–33, 86–87, **30**, 1946, 49–52, 239–242) twelve volumes of tables produced by the N.B.S.M.T.P. (formerly New York M.T.P.) have been described. Items (1) and (2) above are the two latest in the series to be published—bringing the total number of volumes to 21 (there are also smaller tables that have been

published in periodicals, mainly in the *Journal of Mathematics and Physics*). The appearance of (1) provides a suitable occasion for mentioning (3), the first volume produced by the Project, although this is now out of print.

(1) The " Tables of Fractional Powers " given are as follows :

Fifteen-decimal tables.

Table 1 A^x, $A = 2(1)9$; $x = 0(\cdot001)\cdot01(\cdot01)1$.

Table 2, 3 10^x, π^x ; $x = 0(\cdot001)1$.

π^x ; $\pm x = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1(1)12$ (15 figures, $x < 0$)

$\sqrt{2\pi}$, $1/\sqrt{2\pi}$.

Table 4, 5 A^x, $A = \cdot01(\cdot01)\cdot99$, 1000 $A = 101$(primes)997 ; $x = 0(\cdot001)\cdot01(\cdot01)1$.

Table 6–10 $x^{\pm 1/2}$, $x^{\pm 1/3}$, $x^{\pm 2/3}$, $x^{\pm 1/4}$, $x^{\pm 3/4}$; $x = 0(\cdot01)9\cdot99$, also $10^{\pm 1/3}$, $10^{\pm 2/3}$.

Seven-decimal tables.

Table 11 x^a, $a = \cdot01(\cdot01)\cdot99$; $x = 0(\cdot001)X$, $0(\cdot01)1$.

$X = a$ for $a = \cdot01(\cdot01)\cdot3$; $X = \cdot3$ for $a = \cdot3(\cdot01)\cdot6$;

X decreases steadily from $\cdot3$ to $\cdot1$ for $a = \cdot6(\cdot01)\cdot99$.

These tables are undoubtedly of great value, and fill a gap that has been apparent for a long time. They are accompanied by a short introduction which describes the computation of the tables and discusses interpolation; there is also a useful Bibliography of 76 items. No explanation is given of the peculiar limits X in table 11, for which the reviewer can discern no reason. Nor is it clear why the values of $10^{\pm 1/2}$, $10^{\pm 1/4}$, $10^{\pm 3/4}$ are omitted from tables 6, 9 and 10. These are, of course, only minor points.

There is, however, one serious criticism to be made. The arrangement of the arguments x in Tables 1, 4 and 5 is certain to lead to errors in the use of the tables unless unusual and extreme care is taken. The arrangement is such that entries occur in blocks of 5 lines with leading arguments ·01, ·06, ·11, ·16, ... ·46 in the left-hand column, while in the right-hand column they are ·50, ·55, ·60, ·65, ... 0·95 ; (argument ·50 is repeated, while arguments ·00 and 1·00 are omitted!). Thus the arguments ·25 and ·74 are at the same level, while ·25 and ·75 are on levels a line apart. This arrangement is a most unhappy one, and, even more unfortunately, occurs only in Tables 1, 4 and 5, so that its partial occurrence makes it easier to overlook.

Nevertheless, the tables contained in this volume are unique and fundamental, and the book will be needed sooner or later by every computing establishment and by most individual computers.

(2) This is the first of a pair of volumes of " Tables of Spherical Bessel Functions ", that is to say, of tables of the functions

$$\sqrt{\frac{\pi}{2x}} J_\nu(x), \quad 2\nu \text{ an odd integer},$$

also known as Stokes's functions. The first volume gives tables for

$$\pm\nu = \tfrac{1}{2}(1)12\tfrac{1}{2}, \quad x = 0(\cdot01)10(\cdot1)25,$$

$$\pm\nu = 13\tfrac{1}{2}, \qquad x = 0(\cdot01)10(\cdot05)10\cdot5(\cdot1)25.$$

The entries are given to 8, 9 or 10 significant figures for $x \leqslant 10$ and to 7 figures for $x > 10$, except near zeros. Values of δ^2, δ_m^2 or of δ^2 and δ^4 are given wherever

these suffice for interpolation. For regions near $x=0$, such differences are inadequate, and the auxiliary functions

$$x^{-\nu+\frac{1}{2}}\sqrt{\frac{\pi}{2x}}\,J_\nu(x)$$

are tabulated, for $x=0(\cdot01)X$, $\cdot50\leqslant X\leqslant 2\cdot50$; 8 or 10 figures.

Auxiliary tables give interpolation coefficients that may be required when using the tables for every thousandth part of the interval of tabulation.

An Introduction describes "Properties of Bessel Functions", particularly with reference to the methods used for computing the functions. There is no Bibliography.

The functions are of great importance in the study of wave-motion. As remarked by P. M. Morse in his foreword to the volume:

"There are only eleven different Euclidean coordinate systems in which the wave equation can be separated and, unless separation can be carried out, no thoroughgoing theoretical treatment is possible. In the solutions for six of these eleven coordinate systems the ubiquitous Bessel functions are involved. In two—the circular and elliptic cylinder coordinates—Bessel functions of integral order are needed, ... However, in four systems—in spherical, conical, prolate and oblate spheroidal coordinates—Bessel functions are involved having orders equal to one-half an odd integer.... In these four coordinate systems the solutions turn out to involve Bessel functions of half-odd integral orders divided by the square root of the argument."

This volume gives the first tables of these functions apart from a small table given by Morse himself (*Vibration and Sound*, 1936, p. 335) although there are several tables of the straight Bessel functions—see Fletcher, Miller and Rosenhead, *Index of Mathematical Tables*, 1946, Arts. 17·21, 17·421. It may thus be recommended to a wide class of computers and research workers in physical and mathematical problems.

(3) This table of powers was the first table to be prepared and issued by the Mathematical Tables Project. In view of the imminence, at the time it was produced, of B.A. Mathematical Tables, Vol. 9, *Table of Powers* (1940), it was produced in a limited mimeographed edition of foolscap size. The reproduction is consequently inferior to that of later volumes. The volume gives a table of squares, cubes and fourth and fifth powers (together) of numbers from 1 to 1000, followed in succession by tables of sixth and seventh powers; of eighth and ninth powers; and of tenth powers.

All values were compared with the B.A. Table during the production of the latter, without locating any error in the M.T.P. Table, which thus made a worthy start to a most valuable series.

General remarks. All the M.T.P. volumes have been produced by photographic methods from typescript. The use of such methods has great advantages so far as the checking and cost of production are concerned, but the lack of flexibility and the less satisfactory results (though quite good and legible) when compared with those obtained by printing from type are very evident.

A very noticeable point in volumes (1) and (2), as well as in many other M.T.P. volumes, is the small amount of material contained on the average page. This seems due mainly to the fact that typed figures are wider than printed figures of the same height, and sometimes to the use of unsuitable standard forms on which results are typed. It is also probably due to a lack of realisation of the virtues of having as much material to a page as can be arranged without affecting the legibility of the tables. The main virtues seem to be these: (i) Values may be obtained with much less turning of pages;

(ii) More opportunities are available for putting related functions together on an opening ; (iii) Fewer or smaller volumes are needed to cover a given range of tabular values, with consequent saving in book-shelf space, in cost, and in searching for the right volume. This third point is becoming one of major importance to table users, and merits serious consideration. There is much material available, and much more to come, so that it is imperative to avoid undue waste of page-space and of book-shelf space.

If the table of fractional powers had been printed it could have been about half the size, quite easily. Even as it is now, a slight rearrangement to give three columns of values to a page, instead of the two columns that occur on most pages, would have reduced the number of pages by about 30%. The lavish use of space in the table of Spherical Bessel Functions is even more noticeable—the forms on which results are typed are so rarely quite appropriate. Again a 30% or 40% saving of space might easily have been effected, with the possibility of combining this volume with the promised second volume of these functions (to contain functions of order ν, where $\pm\nu = 14\frac{1}{2}(1)\,30\frac{1}{2}$), a much more convenient arrangement.

These remarks have been made in the hope that they may encourage the payment of greater attention to precise details of arrangement, with the emphasis on the point of view of the user, rather than that of the maker, of the table. There is no question that the M.T.P. tables are of the utmost value, and will be needed by everyone making use of the function or functions tabulated.

J. C. P. MILLER.

The Theory of Functions of Real Variables. By L. M. GRAVES. Pp. x, 300. $4. 1946. (McGraw-Hill)

This book, to quote its author, is an exposition of the more fundamental and generally useful parts of real-variable theory. In contrast with many books of similar content and character, where the concept of a real-variable is assumed to be almost intuitive, there are here two introductory chapters, occupying some forty pages, in which this concept is most carefully evolved. In the first of these, the reader is afforded an opportunity of learning the basic principles of symbolic logic (using the Peano-Russell notation) and the elements of the class calculus, which are then applied in the second chapter to the construction of the real number system from Peano's postulates. In spite of its painstaking accuracy, this introduction is formidable and might well have an adverse effect on a beginner desirous of acquainting himself for the first time with the processes of analysis, and the author has therefore been wise enough to suggest, in the preface, that a first reading might well begin with the third chapter, where the elementary theory of point-sets and aggregates is developed.

In the two chapters that follow, the general ideas of continuity and differentiability are fully discussed. A *correct* proof of Rolle's Theorem (a comparative rarity in books of this kind) leads to Taylor's Theorem for one variable, and after the introduction of the partial derivative, the problem of differentiation of a function of several variables is dealt with and the consequent generalisation of Taylor's Theorem is proved.

Chapter VI is devoted to the various definitions and fundamental properties of the Riemann integral and is most valuable as a preliminary to the later chapters concerned with Lebesgue-Stieltjes integrals. This is succeeded by a collection of theorems and results dealing with uniform convergence, multiple limit problems, infinite series and double series and discontinuous functions. Here one feels that, in spite of almost pedantic accuracy, the author fails to succeed in his object, namely to elucidate and disentangle the really involved parts of real-variable analysis, and to present them in a manner appropriate

to the palate of the uninitiated student. One is left with the inevitable feeling that mathematics is a sort of magic, permeable only to genius of such as Peano and Russell, the more regrettable having regard to the fact that it is exactly this attitude that these exponents of logical analysis have striven to destroy. To encourage the more conscientious reader, however, the author has appended a useful and interesting exercise in the form of a questionnaire, in which about thirty definitions and hypotheses have to be linked up with a similar number of statements and conclusions, an innovation that might, to advantage, become a more regular feature of academic textbooks.

Chapters VIII and IX contain material, not usually included in works of this kind, relating to existence theorems on implicit functions and the solutions of ordinary differential equations. In addition to the usual results, many others of a more general character are obtained under relaxed hypotheses with the aid of elementary topology and matrix-theory.

The remaining third of the book is devoted to a study of Lebesgue and Stieltjes integrals. The development of the Lebesgue integral is along the lines originally propounded by F. Riesz, thus making it possible to prove the important convergence and approximation theorems for integrals at the outset, on the basis of only a rudimentary theory of point-sets and measure. The substitution of a generalised measure-function then leads quite naturally to the Lebesgue-Stieltjes integral. This latter, together with its original counterpart, is systematically treated in the concluding chapter.

It is stated in the preface that the book is not intended for isolated study by an inexperienced student, and that the guidance and help of a capable instructor are essential, so that any criticism which is to be fair must therefore be framed with this in mind. Anyone who is familiar with Hardy's *Pure Mathematics* or Bromwich's *Infinite Series* (to quote only two examples of analytical " classics ") will not only marvel in each case at the author's profound knowledge of his subject but his almost uncanny way of foreseeing the novice's difficulties in advance and helping him to overcome them, often by means of what might be described as " exactly the right example at exactly the right time ". This book, though correct to the minutest detail, is in unfortunate contrast to those just mentioned and at times one can almost feel that the author is glorying in his erudition and feels ill-disposed to divulge his secret knowledge to his bewildered reader. On the other hand, one cannot fail to compliment him on the wealth of material that he has presented in the comparatively short space of three hundred pages, nor to pay tribute to him for the inclusion of innumerable references carefully supplied at the conclusion of each chapter.

J. H. PEARCE.

Versicherungsmathematik. By E. ZWINGGI. Pp. 199. Brosch : Fr. Sw. 23 ; geb. : Fr. Sw. 27. 1945. (Birkhäuser, Basel)

The author of this outline of the mathematics of insurance is a Professor at the University of Basle. He states in the Preface that the book has had its origin in the lectures given to students of the subject in their last year at the University. The student of actuarial mathematics in this country will, however, probably find the contents difficult to assimilate. This is due not to any inherent faults in the book but to the differences in the training of British and Continental actuaries. The British student reads for his examinations concurrently with working in an office ; on the Continent he takes a whole-time course at the University and until he has qualified he does not do any practical work. In consequence, the Continental training is apt to be rather more academic than that given in this country.

The book is divided into seven sections, each dealing with one particular aspect of the subject. The sections are, in effect, condensed treatises and are

of varying degrees of excellence. It is impossible to analyse all the sections in a small compass, but the two sections mentioned below are probably the most and least satisfactory of the seven.

Section III deals with assurances on a single life and sets out clearly the various theorems and proofs of formulae necessary for the evaluation of premiums for various forms of policy. Of the several sub-sections those on the principles of invalidity insurance are perhaps the best; many of the propositions will not be found elsewhere in so concise a form.

The last section in the book deals mainly with Graduation and here the author has not quite maintained his high standard. Some of the standard methods (*e.g.* graphic graduation, graduation by summation) are either omitted or dealt with summarily. There is an excellent introduction to analytical processes but too much attention is paid to proofs of formulae which, by the time that he reached his last year, the student should be able to obtain for himself. The section makes no mention of statistical tests of graduation and to this extent fails in its object.

An excellent feature of the book is a list of textbooks, papers and articles on all aspects of the subject, to which students may refer with advantage while studying Prof. Zwinggi's book. The list is fairly comprehensive, although considering the contributions to actuarial literature which have been made by British and American actuaries the proportion of references to their works is surprisingly small. For instance, while Timpe's treatise is one of the two books mentioned on the theory of interest, neither King's *Theory of Finance* nor Todhunter's textbook on *Interest* is deemed worthy of a place. Further, the British reader will wonder why so few of Lidstone's papers in the Journals of the Institute of Actuaries and the Faculty of Actuaries are given: a collection of these papers would themselves form an authoritative textbook on most actuarial subjects.

The above are, however, comparatively minor criticisms of a first-rate outline of the mathematics of insurance. Prof. Zwinggi's book is a welcome addition to the literature of the subject and can be read with advantage by all actuaries who still maintain an interest in the foundations of actuarial mathematics. H. F.

Tools. A Mathematical Sketch and Model Book. By R. C. YATES. $1·60 and postage. (Louisiana State University Press, Baton Range)

This book will act as a tonic on all those who teach geometry and on the greater number who are studying geometry. It may safely be entrusted to any bright student, with the guarantee that it will interest him for weeks, and that it will teach him a good deal of geometry. He will learn, for instance, that with a pair of compasses it is possible to carry out the Euclidean constructions usually performed with a pair of compasses and a straight-edge. He will take delight in and learn much from folding pieces of paper, and he will make linkages from eyelets and pieces of cardboard which will illustrate subtle geometrical theorems and the methods of drawing a straight line. There are nearly 500 drawings in the text, some serving as problems and others as guides to the construction of paper and cardboard models. Section headings are: The straight-edge and modern compasses (Modern Geometry); Dissection of plane polygons; The compasses (Geometry of Mascheroni); Folds and creases (Geometry of paper folding); The straight-edge (Synthetic projective geometry); Line motion linkages (How to draw a straight line); The straight-edge with immovable figure (Geometry of Poncelet-Steiner); The assisted straight-edge; Parallel and angle rulers; Higher tools and quartic systems; General plane linkages.

The author teaches mathematics at West Point Military Academy, N.Y.,

and he has produced a book that is useful, original and stimulating by going back to original sources, some of them British. In fact the book ends with the theorem that any plane algebraic curve can be generated by a linkage, with a reference to the *Proceedings of the London Mathematical Society*. A feeling that this book should have been written here should not diminish our pleasure and gratitude that it has been produced so admirably in the United States. There should be at least one copy in every school which has geometry on its time-table, and all student teachers will want their own copy. D. PEDOE.

Sur les Bases du Groupe Symétrique et les Couples de Substitutions qui Engendrent un Groupe Régulier. By SOPHIE PICCARD. Pp. 223. Fr.s.12. 1946. Mémoires de l'Université de Neuchâtel, Tome Dix-Neuvième. (Paris, Librairie Vuibert)

The first part of this book deals with pairs of permutations which generate the alternating and the symmetric group of n elements. Such a pair is called a base of the group. Various criteria are obtained which allow one to decide whether or not a given pair forms a base. It is shown that apart from a few special cases every permutation is a member of such a base. For $n = 3,4,5,6$ all possible bases are enumerated.

Regular groups of permutations have been studied for a long time. A group of permutations of n objects is regular if it contains exactly n permutations and is also transitive, *i.e.* contains (exactly) one permutation which transforms any of the objects into any other one. The second part of the book is concerned with pairs of permutations which generate such groups.

A great amount of skill and care has been put into this book. Its wealth of results should prove a great help to those who wish to verify results in group theory by examples of special cases. O. T.

Antennae, an Introduction to their Theory. By J. AHARONI. Pp. viii, 265. 25s. 1946. (Oxford)

The scientific design of radio apparatus to secure the maximum output for a given bulk and power input has been the subject of concentrated research during the past ten years, especially the War period 1939–45. In this book the author, who has done original work in radio antennae theory and design, sets out to give an up-to-date mathematical introduction to the subject. There are only three chapters: (1) Antennae and Boundary-Valve Problems, (2) Antennae and Integral Equations, (3) Antennae as Wave Guides. Commencing with Maxwell's equations, there is a discourse on the electromagnetic field, and propagation of various types of wave, *e.g.* transverse electric, and transverse magnetic. The Hertzian dipole, the electromagnetic field due to an oscillating current in a long straight wire, and a double cone (asymmetrical) are discussed. Free electrical oscillations of a sphere and a prolate spheroid follow, the treatment, which is on classical lines, being given in some detail.

In the second chapter similar problems are discussed and the range extended by the method of integral equations. These have the advantage that they are applicable to antennae shapes for which partial differential equations are unsuitable. Nevertheless, the mathematical expressions sometimes occupy a quarter of a page of print. Many curves are given showing the radiation resistance, radiation reactance, and current distribution in straight antennae. The influence of the ground on various properties of a dipole is discussed, and data for design purposes set out in the form of graphs. There is also a short section on direction-finding using frame and vertical aerials. Antennae arrays (groupings of parallel vertical wires) for confining the radiation to a comparatively narrow beam (analogous to a searchlight) are treated, and polar diagrams showing the field distribution are given.

The third chapter is devoted to the subject of antennae as means for " guiding " electromagnetic waves. The analytical work is based upon Maxwell's equations, and not on the ordinary quasi-stationary concept so familiar in transmission line theory, *e.g.* loaded submarine cables. This chapter, which is important from a practical point of view, unfortunately occupies only one-twelfth part of the book. The topics considered are (i) the symmetrical biconical antennae, (ii) the calculation of terminal impedance.

Taken as a whole, this work can be confidently recommended as a reliable guide to the subject of antennae theory. It is not easy to read, because the physical concepts do not seem to have been manipulated with the same dexterity as the mathematics. There appears to be an absence of continuity in the text in the first two chapters, which augments the difficulty of obtaining a comprehensive mental picture of the subject. Although the title says " introduction ", the book is more in the nature of a broad survey suitable for the expert, than an introduction for the general reader who wants to learn the subject.

It is most unfortunate that constructive review comments are always too late, because they cannot be made until after the publication of a book. However, as a matter of general interest the following remarks may be useful. They are in no way intended to detract from the undoubted merits of the book. Some of the symbols used are not standard, *e.g.* q for cross-section instead of A, ν for frequency instead of $f=\omega/2\pi$, n for non-integral orders of a function instead of ν or μ, N for the Y-Bessel function, $Y_n(\eta)$ for $e^{-\frac{1}{2}jn\pi}$, i instead of I for current, and $\dot{i}$ for dI/dt—this being an unessential singularity! The use of $j=\sqrt{(-1)}$ is customary in electrical engineering texts, but i is a better symbol which suggests " imaginary ", while it is standard in mathematical texts. The symbol Q is used for three different things without comment, namely, quantity of electricity, Q factor of a coil, and Legendre's function of the second kind. We also find in connection with the differentiation of $e^{j\omega t}$, that $\partial/\partial t=j\omega$, whereas the rendering should be that $\partial/\partial t$ is replaceable by $j\omega$. It is akin to stating that $p=d/dt$, which is incorrect! On p. 1, in considering an antenna and its supply source, the author says that for maximum power the impedance of the aerial must be matched to the internal impedance of the generator. It should have been stated that maximum power is obtained when the internal resistance of the generator is equal to the total resistance of the aerial, the whole circuit working at unity power factor. This means that the impedances of generator and aerial are conjugate, the circuit reactance being zero.

Most of us have at one time or another written about infinitely long cables, wires, slabs, etc. There is no necessity to be so crude! It is preferable to consider " very long wires such that end effects are negligible ", or to let the diameter of a wire or the thickness of a sheet be very small or $\to 0$. An infinitely long transmission line is usually postulated in the pious hope of avoiding reflection from the far end. For analytical purposes it is adequate to deal with a finite line long enough to avoid serious reflection at an intermediate point, or suppose it to be terminated by its characteristic impedance (where feasible), thereby avoiding reflection and obtaining unidirectional transmission.

Needless to say the printing, format, and general appearance of the book reach a well-known standard of merit which seems to be the monopoly of the Oxford Press. N. W. McL.

Descriptive mathematics. By J. MACLEAN. 2nd edition. Pp. x, 95. 8s. 6d. 1946. (Oxford University Press)

The chief difference between this edition and the first one is in the curtailment of the sections on slide rules and periodic curves respectively. As the

first edition was published in India there may be many who have not seen it. For this reason, and also because it is not obvious from the title what the book is about, it may be well to explain the title and outline the contents in some detail.

The title means that the book deals with those parts of mathematics that are used in describing natural phenomena, and especially the parts which the non-mathematical student will come across during his further studies in any one of the great variety of subjects covered by the headings Science, Engineering and Commerce. So many topics are mentioned that the only way to convey their variety is to make a list of the main ones; these are: Notes on Calculation, Indices, Logarithms and Slide Rule, Graphs, Polynomials, Increments, Integrals, Least Squares, Trigonometrical Functions *via* Polar Coordinates, Periodicity, Solution of Triangles, Solid Geometry, Nomograms, Partial Differentiation, Statistics, Probability, Finite Differences, Glossary of Mathematical Terms.

In spite of the great variety of these topics the author has succeeded well in linking them all to the idea of functional relationship. The book is, in fact, a series of very short, but also very stimulating, essays on the uses of a mathematical function, in its various aspects, for solving problems in science, etc. It is so much condensed that it can be considered as suitable only for either teachers and others already familiar with algebra and calculus, or for students who have textbooks on algebra, calculus, probability, etc., available for reference, and an expert teacher to guide them in using *Descriptive Mathematics* and to tell them where to look for further information on any particular application.

Although it is felt that the uses of this book are rather more limited than the author says was his experience with students in Bombay, nevertheless there is no doubt that it is along the lines advocated by him that progress will be made in adapting our mathematical syllabuses in Universities and Polytechnics to the needs of those not taking mathematics as a subject.

It would be a good thing if there was more experiment by teachers in using some of the methods of presentation suggested, for instance in experimenting to see whether the introduction of calculus through increments, *i.e.* infinitesimals, is more satisfactory with non-specialists than the introduction *via* limits. Teachers would be well advised to get this book as a fruitful source of ideas for stimulating the intellect of the student and for breaking down the taboo surrounding mathematics. It must not, however, be forgotten that much good work has already been done to bridge the gaps between the mathematician and the engineer, the biologist and the economist. H. V. L.

Builders' Calculations. By S. H. GLENISTER. New enlarged edition. Pp. 139. 6s. 1947. (Harrap)

This edition differs from that reviewed on p. 55, Vol. XXX, of the *Mathematical Gazette* by the addition of a few examples to some of the exercises, and a new chapter dealing with easy money calculations.

The whole is bound in a stiff cover against the former limp one, while the cost is increased by one shilling. E. J. A.

Essentials of Business Arithmetic. By H. F. HEMSTOCK. Pp. 342. 6s. 1947. (Harrap)

Arithmetic to some pupils is dull and mechanical. This book, adapted from the American book of the same title by E. M. Kanzer and W. L. Schaaf, is so full of interesting reading matter and illustrative model answers by way of explanation that the subject cannot fail to hold the attention of the keen

student, at the same time giving him a good knowledge of the language and computations used on the financial side of business and commerce.

The opening chapters deal very fully with fundamental mathematical processes, with accuracy and speed in manipulating figures as the guiding principle. Then follow 6 very full chapters or Unit Sections dealing in order with Problems in Trading, including discounts, commissions, profit and loss accounts; Problems in Manufactory, such as Factory Costs, Hour-rates and Piece-work wages; Methods of Payment and Banking; General Business Problems of Rent and Rates, Insurances and Repairs; Business Ownership, Partnerships, Limited Liability Companies, and a final section on Investments. Each section is complete in itself; each contains explanatory paragraphs (which make unusually interesting reading), problems and model answers, and each concludes with a set of " thought questions " followed by a vocabulary.

The exercises have been classified into A. and B. assignments. A. are mechanical and numerical exercises which, the author suggests, " might form the basis for work during the First Year ", while B. are word problems, the more difficult of which are for the Second or Final Year of the Course for which the book is planned. The book is adapted by the author for use in Great Britain from the American book of the same title by E. M. Kanzer and W. L. Schaaf.

It is excellent value at the price, not only for its subject-matter but also for its arrangement and presentable appearance. I would suggest to everyone who is concerned with the organisation of a Commerce Course in any type of School that he should at least see this new Arithmetic. E. J. A.

A New Arithmetic. By A. W. SIDDONS, K. S. SNELL and E. H. LOCKWOOD. I. Pp. vi, 184. 2s. 9d. II. Pp. vi, 210. 3s. III. Pp. vi, 151. 2s. 9d. *Notes and answers.* Pp. 10. 5s. 1947. (Cambridge University Press)

In content and arrangement this work differs from most Arithmetics. It includes a course on Trigonometry which satisfies the requirements of the Alternative Syllabus for the School Certificate Examination. School Trigonometry seems to behave like a planet. It has appeared before in a School Algebra and in a School Geometry; now it completes its orbit in a School Arithmetic. There is a chapter on density and Archimedes' Principle and another on Civic Arithmetic. A section at the end deals with Statistics. There is an early introduction to the use of tables of squares, square roots and reciprocals. Logarithms are developed immediately afterwards. There is an undoubted advantage in the early use of logarithms. It makes possible the use of " natural " numbers, instead of artificial numbers, in such subjects as Mensuration.

Teachers will like this book for several reasons. There are the clear explanations and hints for smoothing out difficulties; even dull children will find it easy to follow the explanations unaided. Here and there we have improvements on current practice. Thus to multiply by $3\frac{1}{7}$ the authors advocate taking 3 times and 1/7 and adding. In multiplication of money the multiplier is put above the money instead of below, which makes it easier for the pupil to follow (why was the same not done in the multiplication of weight on page 14?). The value of rough checks is emphasised throughout.

Great stress is rightly laid on a good style, clear statements, and the accurate use of words; units are insisted upon with all quantities. There is no doubt that the habit of being systematic and accurate has a beneficial influence on pupils whether they are engaged in mathematics or in some other matter. The examples used are varied and have frequent reference to general and topical ideas. Problems are interspersed throughout the book and the aim is

to cultivate mental alertness. At frequent intervals useful hints are given on methods and style which are both helpful and stimulating.

The book, to some extent, follows a concentric scheme. Mensuration commences early and is renewed in four additional chapters spaced through parts 1 to 3. Easy percentages are introduced in part 1 and there are subsequent chapters in parts 2 and 3. Proportion is started as early as chapter 2. The authors are to be congratulated on having nothing to do with Unitary Method. The importance of a correct start in this fundamental subject can not be stressed too much. The authors believe in an early start, a good start, and a thorough follow up, and they suceed admirably in their aim. I am glad that they have made no reference to short cuts on decimalising money. Time spent on learning or teaching such cuts is merely wasted. I am also glad to see the elimination of artificial problems in stocks and shares which have done so much to make the subject disliked. This chapter of Arithmetic belongs to Civics and to that extent the factual aspect should take precedence over "brain teasers" which have done harm to the subject and are out of place.

The authors have given careful consideration to the meaning and value of figures. Nevertheless here and there they underestimate the degree of reliability in an answer obtainable by logarithms and their suggestion that it is preferable to give such an answer to three significant figures is not one which I can support. Leaving out the cases of a high power and an answer which is a difference of values, which should be dealt with on their merits, the accuracy of an answer from logarithmic calculation is much higher than is generally supposed. It can be maintained that (a) it is rare for the error on the 4th figure to be as high as 2; (b) even when such an error occurs a 4-figure answer is much to be preferred to a 3-figure answer. However, as an ounce of experiment is worth a ton of speculation, I invite the authors to evaluate their logarithm and mensuration answers correct to five figures and compare them with the 4-figure logarithm answers. They should find much the same results as I obtained with the 30 examples of Ex. 14e, which is a mixed bag. The number of answers with an error of less than 1 in the 4th figure is 27 (14 are correct to four figures). The remaining three answers are 1 out if the correct answer is given to four figures. By a happy thought, the answer book gives, for the convenience of teachers, the answers to four figures. This will allow teachers to insist on their pupils giving answers to four significant figures.

S. I.

Tutorial Arithmetic. By W. P. Workman. 4th Edition, revised by G. Bossom. Pp. viii, 460. With answers, 8s. 6d.; without answers, 8s. 1947. (University Tutorial Press)

Since *The Tutorial Arithmetic* was first published at the beginning of the century, technique in arithmetical teaching has undergone much development. The syllabus, too, has changed. Topics such as Mensuration and Logarithms are now given much more prominence and others have been pushed into the background or, as in the case of cube root, have disappeared from school textbooks. A revision of the well-known *Tutorial Arithmetic* by W. P. Workman, masterly though it was, was therefore necessary to bring it into line with modern practice. The revision of a work of outstanding merit by another author is seldom successful; nevertheless the present revision has been done with such care and respect for the work of a great teacher that the individuality of the author remains prominent and the reviser has managed skilfully to conceal his own part in the revision.

The test of a teacher is the degree of validity of his principles after a long lapse of time. It is interesting to note how Workman had picked on the best methods many years before experiment proved that they were the best.

Workman had sound judgment and intuition. He maintained that accuracy and speed in fundamental processes are essential, a fact often lost sight of at the present time, and his hints for acquiring these essentials are sound and valuable.

There are several features of this book which make it a valuable addition to the teacher's library. There is the arithmetical theory of fundamentals which is seldom found in school arithmetics. The work abounds in historical details, many of which are absent from the usual books on the History of Mathematics, and at the end there is a section on scales of notation and number theory. The examples at the end are unusual, ingenious and interesting. S. I.

The poetry of mathematics and other essays. By D. E. SMITH. 2nd edition. Pp. vi, 90. $1.25. 1947. *Scripta Mathematica* Library. (*Scripta Mathematica*, 186th Street and Amsterdam Avenue, New York)

The first edition of this book of essays was reviewed in the *Gazette*, XIX, p. 158, and the new edition is a reprint. There are five essays: The Poetry of Mathematics; The Call of Mathematics; Religio Mathematici; Thomas Jefferson and Mathematics; Gaspard Monge, Politician. The two latter are factual, the first three speculative. Professor Smith was a man of wide culture, with an interest in the "humanities" as keen as his interest in mathematics, and his contention that these interests are in fact one is well borne out by his little volume. T. A. A. B.

Four-figure tables. By the late C. GODFREY and A. W. SIDDONS. Pp. 46. 1s. 9d. 1946. (Cambridge University Press)

This new edition of a familiar collection contains the following new material: natural (Napierian) logarithms, e^x, e^{-x}, $\sinh x$, $\cosh x$, all essential in a book of tables for present-day school use. One minor inconsistency should be set right in future editions: on pp. 38–43 we have ϵ, $\log_\epsilon x$, etc., and on p. 45, e, $\log_e N$, etc.

In Mr. Siddons' Presidential Address to the Mathematical Association (*Gazette*, XX, p. 23) he remarks that even in 1900 "four-figure tables were almost unknown to the mathematical class-room". The great change which has come over school mathematics in that respect is largely due, as are so many other changes, to the pioneering work of Godfrey and Siddons.

T. A. A. B.

1542. Mr. Kenneth Lindsay, who moved the Prayer, said that he brought with him the good wishes of a nephew who had won an open scholarship in mathematics at Oxford, and frankly admitted that he owed his success to the teaching of a clergyman. Like a Primrose Leaguer or a Marxist, a clergyman may occasionally show some bias in his teaching of history; but how either politics or theology can be made to interfere with objective teaching of cubes and cosines it is really difficult to imagine.—*The Church Times*, October 26, 1945. [Per Rev. A. F. Mackenzie.]

1543. He [a Professor of Mathematics] read books with titles like "Advanced Propositions on the Higher Aspects of the Differential Calculus", chuckling over them as other men read Wodehouse.—Monica Dickens, *Thursday Afternoons*, p. 79. [Per Prof. A. G. Walker.]

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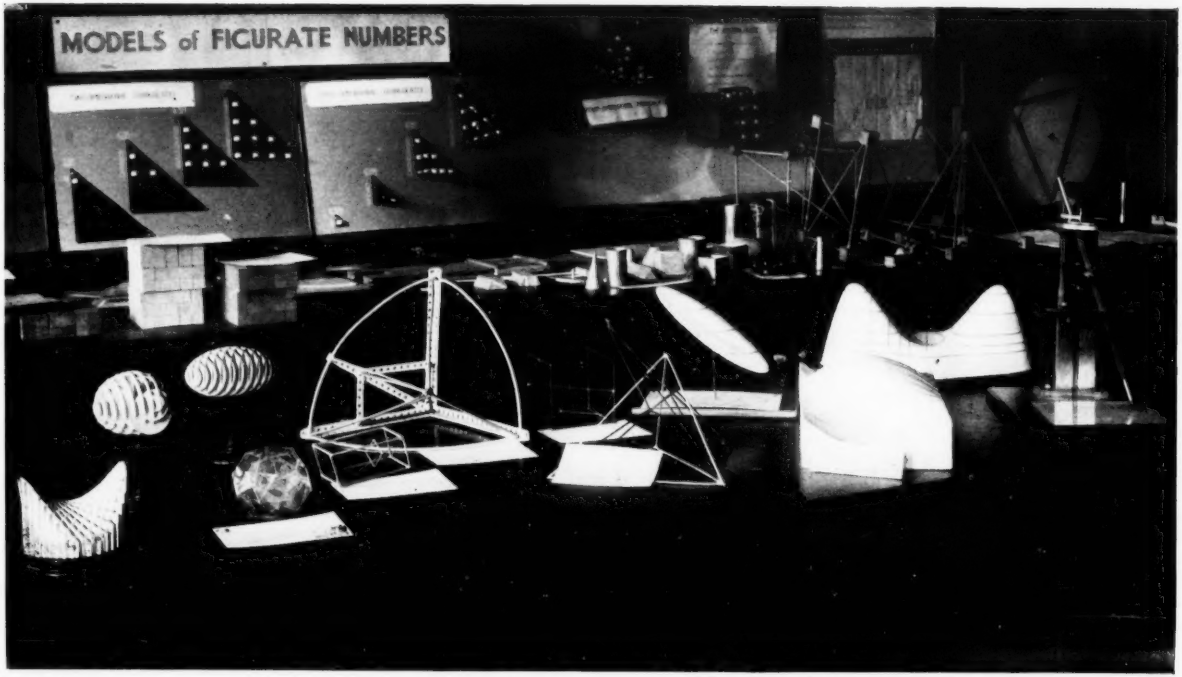

PLATE I.

VISUAL AIDS EXHIBITION, 1947.